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Excess Porteous, Coherent Porteous, and the Hyperelliptic Locus in $\overline{\mathfrak{M}}_3$

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1. Introduction

In [HM, p. 162] the authors consider a family $\pi : X \to B$ of smooth curves of genus 3, not all of which are hyperelliptic, and a map $\sigma : \mathcal{E} \to \mathcal{F}$ of vector bundles (of ranks 3 and 2, respectively) on X. They show that this map fails to be surjective exactly at the hyperelliptic Weierstrass points of hyperelliptic fibers of π. They then use the Thom–Porteous formula for vector bundles to determine an expression for the class of

$$D_1(\sigma) = \{x \in X \mid \operatorname{rank}(\sigma_x) \leq 1\}$$

in the Chow group of X. The authors then use this result to obtain an expression in $\operatorname{Pic}_{\text{fun}}(\mathfrak{M}_3)$ (the group of divisor classes on the moduli stack) for the class of the locus of hyperelliptic curves.

One would like to extend this technique to determine an expression in $\operatorname{Pic}_{\text{fun}}(\overline{\mathfrak{M}}_3)$ for the closure of the locus of hyperelliptic curves. Unfortunately, if one supposes that $\pi : X \to B$ is a family of stable curves of genus 3, then $\mathcal{F}$ will fail to be locally free at singular points of singular fibers of π (see [HM, Sec. 3.F] for details). Harris and Morrison are able to compute this class in $\operatorname{Pic}_{\text{fun}}(\overline{\mathfrak{M}}_3)$ using the method of test curves, but one would still like to extend the original technique to compute the class.

Diaz [D], by constructing a certain blow-up $g : X' \to X$ as well as a map $\sigma' : \mathcal{E}' \to \mathcal{F}'$ of vector bundles on X' that is related to the original map σ, is able to define the degeneracy class for a map of coherent sheaves. The author then applies this process in order to determine an expression in $\operatorname{Pic}_{\text{fun}}(\overline{\mathfrak{M}}_3)$ for the class of the closure of the hyperelliptic locus in $\overline{\mathfrak{M}}_3 \setminus \Delta_1$. Diaz points out that, at singular curves corresponding to general points of Δ_1, not only will $\mathcal{F}$ fail to be locally free at the singular points but also the map σ will have rank ≤ 1 at all points of the elliptic component of the fiber. The author suggests that one could combine the process for determining the degeneracy class of a map of coherent sheaves with the excess Porteous formula found in [F, Exm. 14.4.7] to compute an expression in $\operatorname{Pic}_{\text{fun}}(\overline{\mathfrak{M}}_3)$ for the class of the closure of the hyperelliptic locus in $\overline{\mathfrak{M}}_3$. We will do so in this paper.

To this end, we consider a family $\pi : X \to B$ of smooth, nonhyperelliptic curves degenerating to a general element of Δ_1 and also consider the map $\sigma' : \mathcal{E}' \to \mathcal{F}'$

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