Michigan Math. J. 61 (2012), 239–254

On the Arithmetic Nature of the Values of the Gamma Function, Euler's Constant, and Gompertz's Constant

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1. Introduction

In this paper, we prove some results concerning the arithmetic nature of the values of the Gamma function Γ at rational or algebraic points and for Euler's constant γ. A (completely open) conjecture of Rohrlich and Lang predicts that all polynomial relations between Gamma values over $\mathbb{Q}$ come from the functional equations satisfied by the Gamma function. This conjecture implies the transcendence over $\mathbb{Q}$ of $\Gamma(\alpha)$ at all algebraic nonintegral numbers. But at present, the only known results are the transcendance of $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1/3)$, and $\Gamma(1/4)$ (the last two are algebraically independent of π; see [5]). Using the well-known functional equations satisfied by Γ, we deduce the transcendence of other Gamma values, such as $\Gamma(1/6)$, but not of $\Gamma(1/5)$. Nonetheless, in [7, Thm. 3.3.5] it is proved that the set $\{\pi, \Gamma(1/5), \Gamma(2/5)\}$ contains at least two algebraically independent numbers. In positive characteristic, all polynomial relations between values of the analogue of the Gamma function are known to come from the analogue of the Rohrlich–Lang conjecture (see [1]).

The results proved here are steps in the direction of transcendence results for the Gamma function. We start with a specific quantitative theorem and then prove more general results of a qualitative nature. We define $\log(z)$ and z^α for $z \in \mathbb{C} \setminus (-\infty, 0]$ with the principal value of the argument $-\pi < \arg(z) < \pi$. An important function in the paper is the function

$$\mathcal{G}_\alpha(z) := z^{-\alpha} \int_0^\infty (t+z)^{\alpha-1} e^{-t}\, \mathrm{d}t.$$

For any $\alpha \in \mathbb{C}$, it is an analytic function of z in $\mathbb{C} \setminus (-\infty, 0]$. When $\alpha = 0$ and $z = 1$, $\mathcal{G}_0(1)$ is known as Gompertz's constant (see [6]).

The main result of the paper is the following.

Theorem 1. (i) *For any rational number $\alpha \notin \mathbb{Z}$, any rational number $z > 0$, and any $\varepsilon > 0$, there exists a constant $c(\alpha, \varepsilon, z) > 0$ such that for any $p, q, r \in \mathbb{Z}$, $q \neq 0$, we have*

$$\left|\frac{\Gamma(\alpha)}{z^\alpha} - \frac{p}{q}\right| + \left|\mathcal{G}_\alpha(z) - \frac{r}{q}\right| \geq \frac{c(\alpha, \varepsilon, z)}{H^{3+\varepsilon}}, \tag{1.1}$$

Received July 16, 2010. Revision received March 7, 2011.