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Outer Automorphisms of Algebraic Groups and Determining Groups by Their Maximal Tori

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One goal of this paper is to prove Theorem 20 below, which completes some of the main results in the remarkable paper [PRa1] by Gopal Prasad and Andrei Rapinchuk. For example, combining their Theorem 7.5 with our Theorem 20 gives the following statement.

Theorem 1. *Let G_1 and G_2 be connected, absolutely simple algebraic groups over a number field K that have the same K-isomorphism classes of maximal K-tori. Then*:

(1) *G_1 and G_2 have the same Killing–Cartan type* (*and even the same quasi-split inner form*) *or one has type B_n and the other has type C_n*;
(2) *if G_1 and G_2 are isomorphic over an algebraic closure of K and are not of type A_n for $n \geq 2$, D_{2n+1}, or E_6, then G_1 and G_2 are K-isomorphic.*

This result is mostly proved in [PRa1], except that paper omits types D_{2n} for $2n \geq 4$ in (2). Our Theorem 20 gives a new proof of the $2n \geq 6$ case (treated by Prasad and Rapinchuk in a later paper [PRa2, Sec. 9]) and settles the last remaining case of groups of type D_4. Note that in Theorem 1(2), types A_n, D_{2n+1}, and E_6 are genuine exceptions by [PRa1, 7.6].

Similarly, combining our Theorem 20 with the arguments in [PRa1] implies that their Theorems 4, 8.16, and 10.4 remain true if one deletes "D_4" from their statements—that is, the conclusions of those theorems regarding weak commensurability, locally symmetric spaces, and so on hold also for groups of type D_4.

We mention the following specific result as an additional illustration. For a Riemannian manifold M, write $\mathbb{Q}L(M)$ for the set of rational multiples of lengths of closed geodesics of M.

Theorem 2. *Let M_1 and M_2 be arithmetic quotients of real hyperbolic space $\mathbf{H}^n$ for some $n \not\equiv 1 \bmod 4$. If $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$, then M_1 and M_2 are commensurable* (*i.e., M_1 and M_2 have a common finite-sheeted cover*).

The converse holds with no restriction on n; see [PRa1, Cor. 8.7]. The theorem itself holds for $n = 2$ by [Re]; for $n = 3$ by [CHLRe]; and for $n = 4, 6, 8, \dots$ and $n = 11, 15, 19, \dots$ by [PRa1, Cor. 8.17] (which relies on [PRa2]). The last remaining case, $n = 7$, follows from Theorem 20 (to follow) and arguments as in [PRa1].

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