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Grimm's Conjecture and Smooth Numbers

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1. Introduction

In 1969, C. A. Grimm [8] proposed a seemingly innocent conjecture regarding prime factors of consecutive composite numbers. We begin by stating this conjecture.

Let $n \geq 1$ and $k \geq 1$ be integers. *Suppose* $n+1, \dots, n+k$ *are all composite numbers*; *then there are distinct primes* P_i *such that* $P_i \mid (n+i)$ *for* $1 \leq i \leq k$. That this is a difficult conjecture having several interesting consequences was first pointed out by Erdős and Selfridge [6]. For example, the conjecture implies there is a prime between two consecutive square numbers, something that is out of bounds even for the Riemann hypothesis. In this paper, we will pursue that theme. We will relate several results and conjectures regarding smooth numbers (defined in what follows) to Grimm's conjecture.

To begin, we say that Grimm's conjecture holds for n and k if there are distinct primes P_i such that $P_i \mid (n+i)$ for $1 \leq i \leq k$ whenever $n+1, \dots, n+k$ are all composites. For positive integers $n > 1$ and k, we say that (n,k) *has a prime representation* if there are distinct primes $P_1, P_2, \dots, P_k$ with $P_j \mid (n+j)$ for $1 \leq j \leq k$. We define $g(n)$ to be the maximum positive integer k such that (n,k) has a prime representation. It is an interesting problem to find the best possible upper and lower bounds for $g(n)$. If n' is the smallest prime greater than n, then Grimm's conjecture implies that $g(n) > n' - n$. However, it is clear that $g(2^m) < 2^m$ for $m > 3$.

The question of obtaining lower bounds for $g(n)$ was attacked using methods from transcendental number theory by Ramachandra, Shorey, and Tijdeman [15], who derived

$$g(n) \geq c\left(\frac{\log n}{\log\log n}\right)^3$$

for $n > 3$ and an absolute constant $c > 0$. In other words, for any sufficiently large natural number n, (n,k) has a prime representation if $k \ll (\log n/\log\log n)^3$.

We prove the following theorem.

Theorem 1. (i) *There exists an* $\alpha < \frac{1}{2}$ *such that* $g(n) < n^\alpha$ *for sufficiently large* n.

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