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A Property of Quasi-diagonal Forms

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The aim of this paper is to prove the following result.

Theorem. *Let k be a positive integer and let $F_i \in \mathbb{Z}[\mathbf{x}_i]$ be a form of degree k, where $\mathbf{x}_i$ $(i = 1, 2, \dots)$ are disjoint vectors of variables. Assume that*

$(*)$ *either not all forms are semidefinite of the same sign, or all forms are nonsingular.*

Then there exists a positive integer s_0 such that, for all s, every integer represented by $\sum_{i=1}^{s} F_i(\mathbf{x}_i)$ over $\mathbb{Z}$ is represented by $\sum_{i=1}^{s_0} F_i(\mathbf{x}_i)$ over $\mathbb{Z}$.

If $k = 2$, then the condition $(*)$ can be omitted. J. Szejko has conjectured that the condition $(*)$ is superfluous.

Corollary. *Let k_i be a bounded infinite sequence of positive integers, and let $F_i[\mathbf{x}_i]$ be an infinite sequence of nonsingular forms of degree k_i with the $\mathbf{x}_i$ disjoint. Then there exists a positive integer s_0 such that, for all s, every integer represented by $\sum_{i=1}^{s} F_i(\mathbf{x}_i)$ over $\mathbb{Z}$ is also represented by $\sum_{i=1}^{s_0} F_i(\mathbf{x}_i)$ over $\mathbb{Z}$.*

Remark 1. The assertion is false when $k_i = 2^i$ and $F_i = x_i^{k_i}$ $(i = 1, 2, \dots)$. It may be enough to assume that $\sum_{i=1}^{\infty} \frac{1}{k_i} = \infty$.

Notation. For a given field K and a form $F \in K[x_1, \dots, x_r]$, we use $D(F)$ to denote the Netto discriminant of F—that is, the resultant of $\frac{\partial F}{\partial x_i}$ $(i = 1, 2, \dots, r)$; note that $D(F)$ differs from the true discriminant of F by a constant factor (see [6, p. 434]). Also, $h(F, K)$ is the least h such that $F = \sum_{i=1}^{h} G_i H_i$, where $G_i, H_i \in K[x_1, \dots, x_r]$ are forms of positive degree and $h(F) = h(F, \mathbb{Q})$.

For $a \in \mathbb{Z} \setminus \{0\}$ and p a prime, $\operatorname{ord}_p a$ is the highest exponent e such that $p^e | a$ (i.e., $p^e \| a$); $\operatorname{ord}_p 0 = \infty$. For $\mathbf{x} = [x_1, \dots, x_r] \in \mathbb{Z}^r$, we have $\operatorname{ord}_p \mathbf{x} = \min_{1 \le i \le r} \operatorname{ord}_p x_i$. Finally, $e(x) = \exp\{2\pi i x\}$.

Our proof of the theorem is based on the following series of seventeen lemmas.

Lemma 1. *Let p be a prime, $F \in \mathbb{Z}[\mathbf{x}]$ a form of degree $k = p^\tau k_0$, and $k_0 \in \mathbb{Z} \setminus p\mathbb{Z}$. Let $\gamma = \tau + 2$ if $p = 2$ and $\tau > 0$ and let $\gamma = \tau + 1$ otherwise. If*

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