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On the Johnson Filtration of the Basis-Conjugating Automorphism Group of a Free Group

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1. Introduction

For $n \geq 2$, let F_n be a free group of rank n with basis $x_1, x_2, \dots, x_n$ and let $F_n = \Gamma_n(1), \Gamma_n(2), \dots$ be its lower central series. We denote by $\mathrm{Aut}\, F_n$ the group of automorphisms of F_n. For each $k \geq 0$, let $\mathcal{A}_n(k)$ be the group of automorphisms of F_n that induce the identity on the nilpotent quotient group $F_n/\Gamma_n(k+1)$. The group $\mathcal{A}_n(1)$ is called the IA-automorphism group and is also denoted by IA_n. Then we have a descending filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$, called the *Johnson filtration* of $\mathrm{Aut}\, F_n$.

The Johnson filtration of $\mathrm{Aut}\, F_n$ was originally introduced in 1963 through the remarkable pioneering work by Andreadakis [1], who showed that $\mathcal{A}_n(1), \mathcal{A}_n(2), \dots$ is a descending central series of $\mathcal{A}_n(1)$ and that, for each $k \geq 1$, the graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank. Andreadakis [1] also computed the rank of $\mathrm{gr}^1(\mathcal{A}_n)$. More recently, the independent work of Cohen and Pakianathan [2; 3], Farb [5], and Kawazumi [6] has shown that $\mathrm{gr}^1(\mathcal{A}_n)$ is isomorphic to the abelianization of IA_n. The $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathrm{gr}^k(\mathcal{A}_n) \otimes_{\mathbf{Z}} \mathbf{Q}$ has been determined by Pettet [14] and Satoh [16] for $k = 2$ and 3, respectively. For $k \geq 4$, however, there are few results for the structure of $\mathrm{gr}^k(\mathcal{A}_n)$.

When studying the Johnson filtration, we often face a problem of how to find a generating set of $\mathcal{A}_n(k)$. Although each of the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ is a finitely generated free abelian group, it has not yet been determined whether each of the $\mathcal{A}_n(k)$ is finitely generated or not for $k \geq 2$. Neither do we know whether the abelianization $\mathcal{A}_n(k)^{\mathrm{ab}}$ of $\mathcal{A}_n(k)$ is finitely generated for $k \geq 2$.

In the study of the Johnson filtration of $\mathrm{Aut}\, F_n$, it is also interesting to determine whether or not $\mathcal{A}_n(1), \mathcal{A}_n(2), \dots$ coincides with the lower central series $\mathrm{IA}_n^{(1)}, \mathrm{IA}_n^{(2)}, \dots$ of IA_n. Andreadakis [1] showed that $\mathcal{A}_2(k) = \mathrm{IA}_2^{(k)}$ and $\mathcal{A}_3(3) = \mathrm{IA}_3^{(3)}$. By [2; 3; 5; 6] we have $\mathcal{A}_n(2) = \mathrm{IA}_n^{(2)}$ for $n \geq 3$. Furthermore, in [14] it was shown that $\mathrm{IA}_n^{(3)}$ has finite index in $\mathcal{A}_n(3)$. Andreadakis [1] conjectured that $\mathcal{A}_n(k) = \mathrm{IA}_n^{(k)}$ for any $n \geq 3$ and $k \geq 3$.

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