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Stabilization of Monomial Maps

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Introduction

An important part of higher-dimensional complex dynamics concerns the construction of currents and measures that are invariant under a given meromorphic self-map $f: X \dashrightarrow X$ of a compact complex manifold X. In doing so, it is often desirable that the action of f on the cohomology of X be compatible with iteration; thus, following Sibony [Si] (see also [FoSi]) we call f (algebraically) *stable*.

If f is not stable, we can try to make a bimeromorphic change of coordinates $X' \to X$ such that the induced self-map of X' becomes stable. Understanding when this is possible seems to be a difficult problem. On the one hand, Favre [Fa] showed that stability is not always achievable. On the other hand, it can be achieved for bimeromorphic maps of surfaces [DiF] for a large class of monomial mappings in dimension 2 [F] (more on this below) and for polynomial maps of $\mathbf{C}^2$ [FJ2]. Beyond these classes, very little seems to be known.

In this paper we study the stabilization problem for monomial (or equivariant) maps of toric varieties, extending certain results of Favre to higher dimensions. A toric variety $X = X(\Delta)$ is defined by a lattice $N \cong \mathbf{Z}^m$ and a fan Δ in N. A monomial self-map $f: X \dashrightarrow X$ corresponds to a $\mathbf{Z}$-linear map $\phi: N \to N$. See Sections 1 and 2 for more details.

We work in codimension 1 and say that f is *1-stable* if $(f^n)^* = (f^*)^n$, where f^* denotes the action on the Picard group of X. Geometrically, this means that no iterate of f sends a hypersurface into the indeterminacy set of f [FoSi; Si].

Theorem A. *Let Δ be a fan in a lattice $N \cong \mathbf{Z}^m$, and let $f: X(\Delta) \dashrightarrow X(\Delta)$ be a monomial map. Assume that the eigenvalues of the associated linear map $\phi: N_{\mathbf{R}} \to N_{\mathbf{R}}$ are real and satisfy $\mu_1 > \mu_2 > \cdots > \mu_m > 0$. Then there exists a complete simplicial refinement Δ' of Δ such that $X(\Delta')$ is projective and the induced map $f: X(\Delta') \dashrightarrow X(\Delta')$ is 1-stable.*

Here $N_{\mathbf{R}}$ denotes the vector space $N \otimes_{\mathbf{Z}} \mathbf{R}$. The variety $X' = X(\Delta')$ will not be smooth in general, but it will have at worst quotient singularities. We can pick X' smooth at the expense of replacing f with an iterate (but allowing more general ϕ) as follows.

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