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The Rank of the Second Gaussian Map for General Curves

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Introduction

Let X be a smooth, projective curve of genus g and let $\mathcal{L}$ be a line bundle on X. Consider the product $X \times X$ with the projections p_1, p_2 to the factors and the natural morphism p to the symmetric product $X(2)$. One has $p_*(p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}) = \mathcal{L}^+ \oplus \mathcal{L}^-$, where $\mathcal{L}^\pm$ denotes the invariant and anti-invariant line bundles with respect to the involution $(x, y) \mapsto (y, x)$. One has $H^0(\mathcal{L}^+) \cong \operatorname{Sym}^2 H^0(\mathcal{L})$ and $H^0(\mathcal{L}^-) \cong \wedge^2 H^0(\mathcal{L})$. Restriction to the diagonal of $X(2)$ gives rise to the maps

$$\mu_{\mathcal{L},1} \colon \operatorname{Sym}^2 H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes 2}) \quad \text{and} \quad w_{\mathcal{L},1} \colon \wedge^2 H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes 2} \otimes K_X),$$

where K_X is the canonical bundle of X. Both maps have a well-known geometric meaning. The former is given by considering the map $\phi_{\mathcal{L}} \colon X \to \mathbb{P}^r := \mathbb{P}(H^0(\mathcal{L}))^*$ defined by the complete linear series determined by $\mathcal{L}$ and by pulling forms of degree 2 in $\mathbb{P}^r$ back to X. The latter is given by considering the composition γ of $\phi_{\mathcal{L}}$ with the *Gauss map* of X to the Grassmannian of lines $\mathbb{G}(1, r)$ and by pulling forms of degree 1 in $\mathbb{P}^{\binom{r+1}{2}-1}$ back to X via γ.

The maps $\mu_{\mathcal{L},1}$ and $w_{\mathcal{L},1}$ are the first instances of two hierarchies of maps $\mu_{\mathcal{L},k}$ and $w_{\mathcal{L},k}$, which are defined for all positive integers k and are called by some authors *higher Gaussian maps* of X. They are inductively defined by iterated restrictions to the diagonal of $X(2)$. Precisely, for all $k \geq 2$ one has

$$\mu_{\mathcal{L},k} \colon \ker(\mu_{\mathcal{L},k-1}) \to H^0(\mathcal{L}^{\otimes 2} \otimes K_X^{\otimes 2(k-1)}),$$

$$w_{\mathcal{L},k} \colon \ker(w_{\mathcal{L},k-1}) \to H^0(\mathcal{L}^{\otimes 2} \otimes K_X^{\otimes(2k-1)}).$$

These maps are particularly interesting when $\mathcal{L} \cong K_X$, in which case we will simply denote them as μ_k and w_k. They are both defined at a general point of the moduli space of curves $\mathcal{M}_g$, and it is natural to suppose that they have some modular meaning. Indeed, μ_1 is the codifferential, at the point corresponding to X, of the Torelli map $\tau \colon \mathcal{M}_g \to \mathcal{A}_g$, and Noether's theorem says that μ_1 is surjective if and only if X is nonhyperelliptic.

The map w_1 is called the *Wahl map,* and it is related to important deformation and extendability properties of the canonical image of the curve (cf. [BMé; W]). Because of this, it has been studied by various authors—too many to be quoted

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