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Polyhedral Divisors of Cox Rings

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1. Introduction

Let Z be a $\mathbb{Q}$-factorial projective variety defined over the field of complex numbers such that its divisor class group $\mathrm{Cl}(Z)$ is a lattice that is a free abelian, finitely generated group. We consider the Cox ring of Z,

$$\mathrm{Cox}(Z) = \bigoplus_{D \in \mathrm{Cl}(Z)} \Gamma(Z, \mathcal{O}(D)),$$

with multiplicative structure defined by a choice of divisors whose classes form a basis of $\mathrm{Cl}(Z)$. Our standing assumption in this paper is the finite generation of the $\mathbb{C}$-algebra $\mathrm{Cox}(Z)$. We will call such Z a Mori dream space (or MDS), as it was baptized by Hu and Keel in [HuKe]. We note that a somewhat more general definition of MDS, without the $\mathbb{Q}$ factoriality assumption, was developed by Artebani, Hausen, and Laface [AHL, Thm. 2.3]. However, $\mathbb{Q}$-factoriality of Z is a part of our setup in the present paper.

The $\mathrm{Cl}(Z)$-grading of $\mathrm{Cox}(Z)$ yields an algebraic action of the associated torus $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(Z), \mathbb{C}^*) \cong (\mathbb{C}^*)^{\mathrm{rk}(\mathrm{Cl}(Z))}$ on the affine variety $\mathrm{Spec}(\mathrm{Cox}(Z))$. The variety Z is a GIT quotient of $\mathrm{Spec}(\mathrm{Cox}(Z))$ by the action of this torus. More precisely, a choice of an ample divisor on Z determines an open subset of $\mathrm{Spec}(\mathrm{Cox}((Z))$ such that Z is a good geometric quotient of this set; see [HuKe, Prop. 2.9].

Affine varieties with an algebraic torus action were dealt with by Altmann and Hausen [AlH1], who introduced the notion of polyhedral divisors, or *p-divisors.* Every normal, affine variety X with an algebraic torus action can be described in terms of a polyhedral divisor $\mathcal{D} = \sum_i \Delta_i \otimes D_i$ over its Chow quotient Y [AlH1, Thm. 3.4]. Alternatively, such a p-divisor can be interpreted as a convex, fanwise linear (i.e., piecewise linear and homogeneous, defined on a cone) map from the character lattice M of the torus to $\mathrm{CaDiv}_{\mathbb{Q}}(Y)$; see Section 2.1 for more details. Note that, by abuse of notation, we use the word "Chow quotient" for the normalization of the distinguished component of the inverse limit of the GIT quotients of X (cf. [AlH1, Sec. 6; Hu]).

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