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Cremona Transformations, Surface Automorphisms, and Plane Cubics

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(with an Appendix by Igor Dolgachev)

Introduction

Every automorphism of the complex projective plane $\mathbf{P}^2$ is linear and therefore behaves quite simply when iterated. It is natural to seek other rational complex surfaces—for instance, those obtained from $\mathbf{P}^2$ by successive blowing up—that admit automorphisms with more interesting dynamics. Until recently, very few examples with positive entropy seem to have been known (see e.g. the introduction to [Ca]).

Bedford and Kim [BeK2] found some new examples by studying an explicit family of Cremona transformations—namely, birational self-maps of $\mathbf{P}^2$. McMullen [Mc] gave a more synthetic construction of some similar examples. To this end he used the theory of infinite Coxeter groups, some results of Nagata [N1; N2] about Cremona transformations, and important properties of plane cubic curves. In this paper, we construct many more examples of positive entropy automorphisms on rational surfaces. Whereas [Mc] seeks automorphisms with essentially arbitrary topological behavior, we limit our search to automorphisms that might conceivably be induced by Cremona transformations of polynomial degree 2 (*quadratic transformations* for short). This restriction allows us be more explicit about the automorphisms we find and to make do with less technology, using only the group law for cubic curves (suitably interpreted when the curve is singular or reducible) in place of Coxeter theory and Nagata's theorems.

A quadratic transformation $f\colon \mathbf{P}^2 \to \mathbf{P}^2$ always acts by blowing up three (*indeterminacy*) points $I(f) = \{p_1^+, p_2^+, p_3^+\}$ in $\mathbf{P}^2$ and blowing down the (*exceptional*) lines joining them. Typically, the points and the lines are distinct, but in general they can occur with multiplicity (see Section 1.2). Regardless, f^{-1} is also a quadratic transformation and $I(f^{-1}) = \{p_1^-, p_2^-, p_3^-\}$ consists of the images of the three exceptional lines.

Under certain fairly checkable circumstances, a quadratic transformation f will lift to an automorphism of some rational surface X obtained from $\mathbf{P}^2$ by a finite sequence of point blowups. Namely, suppose there are integers $n_1, n_2, n_3 \in \mathbf{N}$ and a permutation $\sigma \in \Sigma_3$ such that $f^{n_j-1}(p_j^-) = p_{\sigma_j}^+$ for $j = 1, 2, 3$. We assume that

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