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Boundary Behavior of the Kobayashi–Royden Metric in Smooth Pseudoconvex Domains

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1. Introduction and Main Results

In this paper we discuss the problem of the boundary behavior of the Kobayashi–Royden metric (mainly) in the normal direction in smooth bounded pseudoconvex domains. We show two main results. One of the results states in particular that the Kobayashi–Royden metric in the normal direction in some class of smooth bounded pseudoconvex domains is estimated from below by such expressions as $1/d_D^{7/8}(z)$ (where $d_D(z)$ denotes the distance of z from the boundary of D). This improves the result of [Fu], where the author obtained the lower estimate with the exponent 5/6. On the other hand, we demonstrate that a careful study of an example in [FL] shows that the optimal exponent in the lower estimate of the Kobayashi–Royden metric in the normal direction is smaller than 1 (for C^k-smoothness, $k < \infty$). We also specify some obstacles for the rate of the increase in the C^∞ case.

Recall that the Kobayashi–Royden metric has a localization property (see e.g. [G; R]); therefore, we lose no generality in concentrating on domains that are defined globally. Recall also that one of reasons to study the boundary behavior of the Kobayashi–Royden metric is the problem of deciding whether any bounded smooth pseudoconvex domain is Kobayashi complete (see e.g. [JP]). The hope was that the Kobayashi–Royden metric in the normal direction would explode near the boundary as $1/d_D(z)$; it was one of the ideas that were used to show that smooth bounded pseudoconvex domains are Kobayashi complete. However, after many years of uncertainty, an example of Fornæss and Lee [FL] showed that such a lower bound is not valid. More precisely, the example is the following.

Theorem 1 (see [FL]). *For any given increasing sequence $(a_\nu)_\nu$, $a_\nu \to \infty$, of positive numbers, there exist a bounded smooth pseudoconvex domain $D \subset \mathbb{C}^3$ and a decreasing sequence $(\delta_\nu)_\nu$ with $\delta_\nu \to 0$ such that*

$$\kappa_D(P_{\delta_\nu}; n) \le \frac{1}{(a_\nu \delta_\nu)},$$

where P is a suitable point from ∂D, $P_{\delta_\nu} = P - \delta_\nu n$, and n is the unit outward normal vector to ∂D at P.

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