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On the Greatest Common Divisor of a Number and Its Sum of Divisors

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1. Introduction

A natural number n is called *perfect* if $\sigma(n) = 2n$ and *multiply perfect* whenever $\sigma(n)$ is a multiple of n. In 1956, Erdős published improved upper bounds on the counting functions of the perfect and multiply perfect numbers [2]. These estimates were soon superseded by a theorem of Wirsing [15] (Theorem B in this paper), but Erdős's methods remain of interest because they are applicable to more general questions concerning the distribution of $\gcd(n, \sigma(n))$. Erdős [2] describes some applications of this type but omits the proofs. In this paper, we prove corrected versions of his results and establish some new results in the same direction.

For real numbers $x \ge 1$ and $A \ge 1$, put

$$G(x, A) := \#\{n \le x : \gcd(n, \sigma(n)) > A\}.$$

Theorem 1.1. *Let $\beta > 0$. If $x > x_0(\beta)$ and $A > \exp((\log\log x)^{\beta})$, then $G(x, A) \le x/A^c$, where $c = c(\beta) > 0$.*

This is (more or less) Theorem 3 of [2], except that Erdős assumes instead that $A > (\log x)^{\beta}$. After stating Theorem 3, Erdős claims that his result is best possible in that, if A grows slower than any power of $\log x$, then one does not save a fixed power of A in the estimate for $G(x, A)$. Theorem 1.1 shows that this assertion is incorrect. Our next result shows that Theorem 1.1 is best possible in the sense Erdős intended.

Theorem 1.2. *Let $\beta = \beta(x)$ be a positive real-valued function of x satisfying $\beta(x) \to 0$ as $x \to \infty$. Let $\varepsilon > 0$. If x is sufficiently large (depending on ε and the choice of function β) and $2 \le A \le \exp((\log\log x)^{\beta})$, then $G(x, A) \ge x/A^{\varepsilon}$.*

For large values of A, one may deduce a stronger upper bound on $G(x, A)$ than that of Theorem 1.1 from the following estimate for the mean of $\gcd(n, \sigma(n))$.

Theorem 1.3. *For each $x \ge 3$, we have*

$$\sum_{n \le x} \gcd(n, \sigma(n)) \le x^{1 + c_1/\sqrt{\log\log x}},$$

where c_1 is an absolute positive constant.

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