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On Volumes along Subvarieties of Line Bundles with Nonnegative Kodaira–Iitaka Dimension

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1. Introduction

We study the restricted volume along subvarieties of line bundles with nonnegative Kodaira–Iitaka dimension. Our main interest is to compare it with a similar notion defined in terms of the asymptotic multiplier ideal sheaf, with which it coincides in the big case. We shall prove that the former is nonzero if and only if the latter is. We then study inequalities between them and prove that if they coincide on every very general curve then the line bundle must have zero Kodaira–Iitaka dimension or be big.

Let X be a smooth projective variety and L a divisor or a line bundle on X with nonnegative Kodaira–Iitaka dimension: $\kappa(L) \geq 0$. Let $V \subset X$ be a subvariety of $\dim V = d > 0$ such that $V \not\subset \mathrm{SBs}(L)$, where $\mathrm{SBs}(L) := \bigcap_{m>0} \mathrm{Bs}|mL|$ is the stable base locus. We denote by $H^0(X|V, mL) = \mathrm{Image}[H^0(X, mL) \to H^0(V, mL)]$ the image of restriction maps. The *restricted volume* of L along V is defined to be

$$\mathrm{vol}_{X|V}(L) = \limsup_{m\to\infty} \frac{h^0(X|V, mL)}{m^d/d!}.$$

Similary, we define the *reduced volume* of L along V as follows:

$$\mu(V, L) = \limsup_{m\to\infty} \frac{h^0(V, \mathcal{O}_V(mL) \otimes \mathcal{J}(\|mL\|)|_V)}{m^d/d!}.$$

Here $\mathcal{J}(\|mL\|) = \mathcal{J}(X, \|mL\|)$ is the asymptotic multiplier ideal sheaf of mL for every positive integer m [L, 11.1.2]. When L is big, $\mu(V, L) = \mathrm{vol}_{X|V}(L) > 0$ for any $V \not\subset \mathrm{NAmp}(L)$ [ELMNP3, 2.13; T3, 3.1], where

$$\mathrm{NAmp}(L) := \bigcap_{m>0} \mathrm{SBs}(mL - A)$$

for any given ample divisor A on X and is called the nonample locus of L (in [L, 10.3.2], this is denoted by $\mathbf{B}_+(L)$ and called the augmented base locus). In the big case, the restricted volume has played an important role in the proof of the boundedness of pluricanonical maps (cf. [HMc; T3; Ts2]) and the topic has been systematically studied by Ein, Lazarsfeld, Mustaţă, Nakamaye, and Popa in [ELMNP1; ELMNP2; ELMNP3; L]. On the other hand, very little is known in

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