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Inequivalent Embeddings of the Koras–Russell Cubic 3-Fold

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1. Introduction

The Koras–Russell cubic 3-fold is the hypersurface X of the complex affine space $\mathbb{A}^4 = \operatorname{Spec}(\mathbb{C}[x, y, z, t])$ defined by the equation

$$P = x + x^2 y + z^2 + t^3 = 0.$$

It is well-known that X is an affine contractible smooth 3-fold that is not algebraically isomorphic to an affine 3-space. The main result of this paper is to show that there exists another hypersurface Y of $\mathbb{A}^4$ that is isomorphic to X but such that there exists no automorphism of the ambient 4-space that restricts to an isomorphism between X and Y. In other words, the two hypersurfaces are inequivalent. In order to prove this result, we give a description of the automorphism group of X. It is shown that all algebraic automorphisms of X extend to automorphisms of $\mathbb{A}^4$, which implies that every automorphism of X fixes the point $(0,0,0,0) \in X$.

More specifically, we will study certain properties of the Koras–Russell cubic 3-fold. The point of view comes from the elementary remark that this 3-fold can be interpreted as a 1-parameter family of Danielewski hypersurfaces. A *Danielewski hypersurface* is a subvariety of $\mathbb{A}^3 = \operatorname{Spec}(\mathbb{C}[x, y, z])$ defined by an equation of the form $x^n y = q(x, z)$, where n is a nonzero natural number and $q(x, z) \in \mathbb{C}[x, z]$ is a polynomial such that $q(0, z)$ is of degree at least 2. Such hypersurfaces have been studied by the authors in [4], [13], and [14]. This interpretation allows us to deduce results similar to the ones for Danielewski hypersurfaces for this 3-fold.

An important question in affine algebraic geometry asks whether every embedding of complex affine k-space $\mathbb{A}^k$ in $\mathbb{A}^n$, where $k < n$, is rectifiable—in other words, is equivalent to an embedding as a linear subspace. The Abyhankar–Moh–Suzuki theorem shows that the answer is "yes" if $n = 2$ [1; 17] and, by a general result proved independantly by Kaliman [8] and Srinivas [16], if $n \geq 2k + 2$ then the answer is also affirmative. However, all other cases remain open.

Here we are interested in the case of embeddings of hypersurfaces. It is easy to find affine varieties of dimension n admitting nonequivalent embeddings into $\mathbb{A}^{n+1}$. For example, the punctured line $\mathbb{A}^1 \setminus \{0\}$ has many nonequivalent embeddings in $\mathbb{A}^2$. For each $n \in \mathbb{N}$, let $P_n = x^n y - 1$. The subvariety defined by the zero

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