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On the Debarre–de Jong and Beheshti–Starr Conjectures on Hypersurfaces with Too Many Lines

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1. Introduction

Let K be an algebraically closed field of characteristic 0. Write X_{sing} for the singular points of a variety X, $\mathbb{P}^n = K\mathbb{P}^n$, and $\mathbb{G}(\mathbb{P}^1, \mathbb{P}^n) = G(2, n+1)$ for the Grassmannian.

The following conjecture essentially states that if $X^{n-1} \subset \mathbb{P}^n$ has "too many" lines then, for any point $x \in X$ that has (too many) lines going through it, one of the lines through x will contain a singular point of X.

Conjecture 1.1. *Let $X^{n-1} \subset \mathbb{P}^n$ be a hypersurface of degree $d \geq n$ and let $\mathbb{F}(X) \subset G(2, n+1)$ denote the Fano scheme of lines on X. Let $B \subset \mathbb{F}(X)$ be an irreducible component of dimension at least $n-2$. Let $\mathcal{I}_B := \{(x, E) \mid x \in X, E \in B, x \in \mathbb{P}E\}$, and let π and ρ denote (respectively) the projections to X and B. Let $X_B = \pi(\mathcal{I}_B) \subseteq X$ and let $\tilde{\mathcal{C}}_x = \pi\rho^{-1}\rho\pi^{-1}(x)$.*

Then, for all $x \in X_B$, $\tilde{\mathcal{C}}_x \cap X_{\text{sing}} \neq \emptyset$.

If we take hyperplane sections in the case $d = n$, then Conjecture 1.1 would imply the following, which was conjectured independently by Debarre and de Jong.

Conjecture 1.2 (Debarre–de Jong conjecture). *Let $X^{n-1} \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \leq n$. Then the dimension of the Fano scheme of lines on X equals $2n - d - 3$.*

Our conjecture extends to smaller degrees as follows.

Conjecture 1.3. *Let $X^{n-1} \subset \mathbb{P}^n$ be a hypersurface of degree $n - \lambda$. Let $B \subset \mathbb{F}(X)$ be an irreducible component of dimension $n-2$ with $\mathcal{I}_B, X_B, \dots$ as before. If* $\operatorname{codim}(X_B, X) \geq \lambda$ *and $\mathcal{C}_x$ is reduced for general $x \in X_B$, then for all $x \in X_B$, $\tilde{\mathcal{C}}_x \cap X_{\text{sing}} \neq \emptyset$.*

The cases $X_B = X$ and $\operatorname{codim}(X_B, X) = n/2$ are known; for example, they appear in Debarre's unpublished notes containing Conjecture 1.2. In [9], Harris,

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