Michigan Math. J. 59 (2010), 535–572

# Extremal Rational Elliptic Threefolds

Arthur Prendergast-Smith

An elliptic fibration is a proper morphism $f: X \to Y$ of normal projective varieties whose generic fibre $E$ is a regular curve of genus 1. The Mordell–Weil rank of such a fibration is defined to be the rank of the abelian group $\operatorname{Pic}^0 E$ of degree-0 line bundles on $E$. In particular, $f$ is called *extremal* if its Mordell–Weil rank is 0.

The simplest nontrivial elliptic fibration is a rational elliptic surface $f: X \to \mathbf{P}^1$. There is a complete classification of extremal rational elliptic surfaces due to Miranda and Persson in characteristic 0 [14] and to Lang in positive characteristic [12; 13]. (See also Cossec and Dolgachev [4, Sec. 5.6].) The purpose of the present paper is to produce a corresponding classification of a certain class of extremal rational elliptic threefolds. For reference, the results are shown in Table 1.

Let us say a bit more about exactly which objects we are classifying. It is a classical fact that any rational elliptic surface is the blowup of $\mathbf{P}^2$ at the base locus (a 0-dimensional subscheme of degree 9) of a pencil of cubic curves. This description allows one to compute the Mordell–Weil rank in terms of reducibility properties of curves in the pencil [17, Thm. 5.2]. In dimension 3, the analogous situation is to consider a net (2-dimensional linear system) of quadric surfaces in $\mathbf{P}^3$. The base locus of such a net is a 0-dimensional subscheme of degree 8. We will see in what follows that, under a certain nondegeneracy assumption on the net, blowing up at the base locus gives an elliptic fibration $f: X \to \mathbf{P}^2$, and then we can compute the Mordell–Weil rank of $f$ in terms of reducibility properties of quadrics in the net. To exploit this, we will consider in this paper only elliptic threefolds obtained by blowing up the base locus of a net of quadrics in $\mathbf{P}^3$. Table 1 gives a list of all nets of quadrics (up to projective equivalence) that give rise to extremal elliptic threefolds in this way.

The classification may be of interest for several reasons. First, it is a natural counterpart of the results of Miranda–Persson and Lang on extremal rational elliptic surfaces. It is perhaps surprising to see that the situation for threefolds, in which the classification contains only a small finite number of cases, is simpler than that for surfaces. Second, the method of proof uses the theory of root systems in an essential way. This gives a further demonstration of the strong connection—elaborated in [6] and [4]—between root systems and configurations of points in projective space. Finally, the classification provides “test specimens” for the cone

Received March 13, 2009. Revision received August 13, 2009.