Michigan Math. J. 59 (2010), 505–515

Nonlinearity of Morphisms in Non-Archimedean and Complex Dynamics

Yûsuke Okuyama

Dedicated to the memory of Professor Juha Heinonen

1. Introduction

One of the aims of this paper is extending the fundamental Cremer theorem from the iteration theory of one complex variable to the setting of higher-dimensional dynamics over more general valued-fields, not necessarily $\mathbb{C}$. We note that analytic function theory over such fields was already well prepared in the fundamental work [A] around 1960.

Let K be a commutative *algebraically closed* field that is complete and nontrivial with respect to an absolute value (or valuation) $|\cdot|$. Then $|\cdot|$ is said to be *non-Archimedean* if, for all $z, w \in K$, $|z - w| \le \max\{|z|, |w|\}$. Otherwise, $|\cdot|$ is said to be *Archimedean,* in which case K is topologically isomorphic to $\mathbb{C}$ (with Hermitian norm). We extend $|\cdot|$ to K^ℓ ($\ell \in \mathbb{N}$) as the maximum norm $|Z| = |Z|_\ell = \max_{j=1,\dots,\ell}|z_j|$ for $Z = (z_1, \dots, z_\ell)$. We consider the polydisk

$$\bar{P}(Z_0, r) = \bar{P}^\ell(Z_0, r) := \{Z \in K^\ell;\ |Z - Z_0| \le r\}$$

for $Z_0 \in K^\ell$ and $r > 0$. The extended $|\cdot|_\ell$ is non-Archimedean if and only if the original $|\cdot|_1$ is also, and in this case

$$\operatorname{int} \bar{P}(Z_0, r) = \bar{P}(Z_0, r).$$

We denote the origin in K^ℓ by $O = O_\ell$. In the Archimedean case $K = \mathbb{C}$, $\mathbb{C}^\ell$ also has the Hermitian norm $\|\cdot\| = \|\cdot\|_\ell$ ($\asymp |\cdot|_\ell$ uniformly).

Let $\pi\colon K^{n+1} \setminus \{O\} \to \mathbb{P}^n(K)$ be the canonical projection. Set the integer $\ell(n) = \binom{n+1}{2}$ so that $\bigwedge^2 K^{n+1} \cong K^{\ell(n)}$ (cf. [Ko, Sec. 8.1]). We equip $\mathbb{P}^n(K)$ with the *chordal distance* $[z, w]$ between $z, w \in \mathbb{P}^n(K)$, defined as

$$[z, w] := \begin{cases} \dfrac{|Z \wedge W|_{\ell(n)}}{|Z|_{n+1}|W|_{n+1}} \le 1 & (|\cdot| \text{ is non-Archimedean}), \\[2ex] \dfrac{\|Z \wedge W\|_{\ell(n)}}{\|Z\|_{n+1}\|W\|_{n+1}} \le 1 & (|\cdot| \text{ is Archimedean}), \end{cases} \tag{1.1}$$

where $Z \in \pi^{-1}(z)$ and $W \in \pi^{-1}(w)$. For $z_0 \in \mathbb{P}^n(K)$ and $r > 0$, we consider the ball

$$\bar{B}(z_0, r) := \{z \in \mathbb{P}^n(K);\ [z, z_0] \le r\}.$$

Received October 27, 2008. Revision received May 27, 2010.