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An Elementary Proof of the Cross Theorem in the Reinhardt Case

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1. Introduction and Main Result

The problem of continuation of separately holomorphic functions defined on a cross has been investigated in several papers (e.g., [B; S1; S2; AkR; Za; S3; Sh; NS; NZ1; NZ2; N; AZ; Z]) and may be formulated in the form of the following *cross theorem*.

Theorem 1.1. *Let $D_j \subset \mathbb{C}^{n_j}$ be a domain of holomorphy and let $A_j \subset D_j$ be a locally pluriregular set, $j = 1, \dots, N$, $N \ge 2$. Define the* cross

$$\boldsymbol{X} := \bigcup_{j=1}^{N} A_1 \times \cdots \times A_{j-1} \times D_j \times A_{j+1} \times \cdots \times A_N.$$

Let $f : \boldsymbol{X} \to \mathbb{C}$ be separately holomorphic*—that is, for any $(a_1, \dots, a_N) \in A_1 \times \cdots \times A_N$ and $j \in \{1, \dots, N\}$, the function*

$$D_j \ni z_j \longmapsto f(a_1, \dots, a_{j-1}, z_j, a_{j+1}, \dots, a_N) \in \mathbb{C}$$

is holomorphic. Then f extends holomorphically to a uniquely determined function $\hat f$ on the domain of holomorphy

$$\hat{\boldsymbol{X}} := \Big\{(z_1, \dots, z_N) \in D_1 \times \cdots \times D_N : \sum_{j=1}^{N} h^*_{A_j, D_j}(z_j) < 1\Big\}, \qquad (*)$$

where $h^*_{A_j,D_j}$ is the upper regularization of the relative extremal function h_{A_j,D_j}, *$j = 1, \dots, N$.*

Recall that $h_{A,D} := \sup\{u \in \mathcal{PSH}(D) : u \le 1, u|_A \le 0\}$.

Observe that in the case where A_j is open, $j = 1, \dots, N$, the cross $\boldsymbol{X}$ is a domain in $\mathbb{C}^n$ with $n := n_1 + \cdots + n_N$. Moreover, by the classical Hartogs lemma, every separately holomorphic function on $\boldsymbol{X}$ is simply holomorphic. Consequently, the formula $(*)$ is nothing more than a description of the envelope of holomorphy of $\boldsymbol{X}$. Thus, it is natural to conjecture that in this case the formula $(*)$ may be obtained without the cross theorem machinery. Unfortunately, we do not know of any such simplification.

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