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On the Hypersurface of Lüroth Quartics

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Introduction

In his celebrated paper [18], Lüroth proved that a nonsingular quartic plane curve containing the ten vertices of a complete pentalateral contains infinitely many such 10-tuples. This implies that such curves, called *Lüroth quartics,* fill an open set of an irreducible, SL(3)-invariant hypersurface $\mathcal{L} \subset \mathbb{P}^{14}$. In his short paper [19], Morley computed the degree of the Lüroth hypersurface $\mathcal{L}$ by introducing some interesting ideas that seem to have been forgotten, maybe because a few arguments are somehow obscure. In this paper we put Morley's result and method on a solid foundation by reconstructing his proof as faithfully as possible. The main result is the following.

Theorem 0.1. *The Lüroth hypersurface $\mathcal{L} \subset \mathbb{P}^{14}$ has degree 54.*

Morley's proof uses the description of plane quartics as branch curves of the degree-2 rational self-maps of $\mathbb{P}^2$ called *Geiser involutions.* Every such involution is determined by the linear system of cubics having as base locus a 7-tuple of distinct points $Z = \{P_1, \dots, P_7\}$; let's denote by $B(Z) \subset \mathbb{P}^2$ the corresponding quartic branch curve. Morley introduces a closed condition on the space of such 7-tuples given by the vanishing of the Pfaffian of a natural skew-symmetric bilinear form between conics associated to each such Z. By this procedure one obtains an irreducible polynomial $\Psi(P_1, \dots, P_7)$ that is multihomogeneous of degree 3 in the coordinates of the points $P_1, \dots, P_7$ and skew-symmetric with respect to their permutations. We call Ψ *the Morley invariant*. The symbolic expression of Ψ is related to $\mathbb{P}^2_{\mathbb{Z}/2\mathbb{Z}}$, classically known as *the Fano plane* (see Section 4).

Then Morley proceeds to prove that the nonsingular quartics $B(Z)$ corresponding to the 7-tuples Z for which the Morley invariant vanishes are precisely the Lüroth quartics. This step of the proof uses a result of Bateman [2], which gives an explicit description of an irreducible 13-dimensional family of configurations Z such that $B(Z)$ is Lüroth: Morley shows that the Bateman configurations are precisely those making Ψ vanish. In order to gain control on the degree of $\mathcal{L}$, one must consider the full locus of configurations Z such that $B(Z)$ is a Lüroth

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