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On the Beurling–Ahlfors Transform's Weak-type Constant

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1. Introduction

The Beurling–Ahlfors transform, denoted by S, is defined on $L^p(\mathbb{C})$, $1 \le p < \infty$, by

$$Sf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\,dw,$$

where dw is the 2-dimensional Lebesgue measure and the integral is understood as a Cauchy principal value. By the theory of Calderón and Zygmund this is a bounded operator on $L^p(\mathbb{C})$ for $1 < p < \infty$. In fact one can show the Fourier multiplier associated with the operator is $\bar{\xi}/\xi$, and so it is an isometry on $L^2(\mathbb{C})$ by Plancharel's theorem. Its norm on the other L^p spaces is unknown and is an area of interest, especially in the field of quasiconformal mappings as

$$S(\bar{\partial} f) = \partial f$$

gives a connection between the z and $\bar{z}$ derivatives. The well-known Iwaniec conjecture [4] asserts that

$$\|S\|_p = p^* - 1 := \max\{p, p/(p-1)\} - 1.$$

The current best estimate of

$$\|S\|_p \le 1.575(p^* - 1)$$

is due to Bañeulos and Janakiraman [2].

For a function f on a measure space (X,μ) we define the weak "norm" of f as

$$\|f\|_w := \sup_{\lambda>0} \mu(|f| \ge \lambda)\lambda.$$

For an operator T defined on $L^1(X,\mu)$, but not necessarily bounded, define

$$\|T\|_w := \sup_{f\in L^1,\, f\ne 0} \frac{\|Tf\|_w}{\|f\|_1}.$$

An $L^1(X,\mu)$ bounded operator T has $\|T\|_w < \infty$ by Chebyshev's inequality, but an operator with $\|T\|_w < \infty$ is not necessarily bounded in $L^1(X,\mu)$. If $\|T\|_w$ is finite, we say that T is *weak-type bounded* with constant $\|T\|_w$.

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