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The Automorphism Group of the Free Group of Rank 2 Is a CAT(0) Group

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1. Introduction

A *CAT(0) metric space* is a proper complete geodesic metric space in which each geodesic triangle with side lengths a, b, and c is "at least as thin" as the Euclidean triangle with side lengths a, b, and c (see [5] for details). We say that a finitely generated group G is a *CAT(0) group* if G may be realized as a cocompact and properly discontinuous subgroup of the isometry group of a CAT(0) metric space X. Equivalently, G is a CAT(0) group if there exists a CAT(0) metric space X and a faithful geometric action of G on X. It is perhaps not standard to require that the group action be faithful, a point we address in Remark 1.

For each integer $n \geq 2$, we write F_n for the free group of rank n and B_n for the braid group on n strands.

In [3], Brady exhibited a subgroup $H \leq \mathrm{Aut}(F_2)$ of index 24 that acts faithfully and geometrically on a CAT(0) 2-complex. In subsequent work [4], the same author showed that B_4 acts faithfully and geometrically on a CAT(0) 3-complex. It follows that $\mathrm{Inn}(B_4)$ acts faithfully and geometrically on a CAT(0) 2-complex X_0 (this fact is explained explicitly by Crisp and Paoluzzi in [8]). Now, $\mathrm{Inn}(B_n)$ has index 2 in $\mathrm{Aut}(B_n)$ [10], and $\mathrm{Aut}(F_2)$ is isomorphic to $\mathrm{Aut}(B_4)$ [16, 10]; thus the result in the title of this paper is proved if we exhibit an extra isometry of X_0 that extends the faithful geometric action of $\mathrm{Inn}(B_4)$ to a faithful geometric action of $\mathrm{Aut}(B_4)$. We do this in Section 2.

In the language of [14], X_0 is a systolic simplicial complex. By [14, Thm. 13.1], a group that acts simplicially, properly discontinuously, and cocompactly on such a space is biautomatic. Since the action of $\mathrm{Aut}(F_2)$ provided here is of this type, it follows that $\mathrm{Aut}(F_2)$ is biautomatic.

Our results reinforce the striking contrast between those properties enjoyed by $\mathrm{Aut}(F_2)$ and those enjoyed by the automorphism groups of finitely generated free groups of higher ranks. We can now say that $\mathrm{Aut}(F_2)$ is a CAT(0) group, that it is a biautomatic group, and that it has a faithful linear representation [9; 16]; while $\mathrm{Aut}(F_n)$ is neither a CAT(0) group [12] nor a biautomatic group [6], and it does not have a faithful linear representation [11] whenever $n \geq 3$.

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