Michigan Math. J. 59 (2010), 269–281

Complete Intersection Points on General Surfaces in $\mathbb{P}^3$

Enrico Carlini, Luca Chiantini, & Anthony V. Geramita

1. Introduction

A recurrent theme in classical projective geometry is the study of special subvarieties of some given family of varieties. For example: How many isolated singular points can a surface of degree d in $\mathbb{P}^3$ have? Under what conditions do members of a certain family of varieties contain rational curves, or contain a linear space of some positive dimension? Other examples of similar questions can easily be provided by the reader.

The study of the special case of complete intersection subvarieties of hypersurfaces in $\mathbb{P}^n$ has been the subject of a great deal of research. It was known to Severi [S] that, for $n \geq 4$, the only complete intersections of codimension 1 on a general hypersurface are obtained by intersecting that hypersurface with another. Indeed, the problem can be seen as a particular instance of the general problem of determining subvarieties of general hypersurfaces in $\mathbb{P}^n$ (see e.g. [GrH; Gro; L]).

In [CChG] we proposed a new approach to the problem of studying complete intersection subvarieties of hypersurfaces. This approach uses a mix of projective geometry and commutative algebra and is more direct than the usual methods for addressing the general problem. With our approach we were able to give a complete description of the situation for complete intersections of codimension r in $\mathbb{P}^n$ that lie on a general hypersurface of degree d whenever $2r \leq n+2$. The main result of [CChG] is as follows.

Theorem 1.1. *Let $X \subset \mathbb{P}^n$ be a generic degree-d hypersurface, where $n, d > 1$. Then X contains a complete intersection of type $(a_1, \dots, a_r)$, with $2r \leq n+2$ and the a_i all less than d, in the following (and only in the following) instances.*

- *$n = 2$: then $r = 2$, d is arbitrary, and a_1 and a_2 can assume any value less than d.*
- *$n = 3$, $r = 2$: for $d \leq 3$ we have that a_1 and a_2 can assume any value less than d.*
- *$n = 4$, $r = 3$: for $d \leq 5$ we have that a_1, a_2, and a_3 can assume any value less than d.*
- *$n = 6$, $r = 4$ or $n = 8$, $r = 5$: for $d \leq 3$ we have that $a_1, \dots, a_r$ can assume any value less than d.*

Received November 13, 2008. Revision received April 27, 2009.