Michigan Math. J. 59 (2010), 243–267

Singular Loci of Grassmann–Hibi Toric Varieties

J. Brown & V. Lakshmibai

Introduction

Let K denote the base field, which we assume to be algebraically closed and of arbitrary characteristic. Given a distributive lattice $\mathcal{L}$, let $X(\mathcal{L})$ denote the affine variety in $\mathbb{A}^{\#\mathcal{L}}$ whose vanishing ideal is generated by the binomials $X_\tau X_\varphi - X_{\tau\vee\varphi}X_{\tau\wedge\varphi}$ in the polynomial algebra $K[X_\alpha, \alpha \in \mathcal{L}]$ (here, $\tau \vee \varphi$ (resp. $\tau \wedge \varphi$) denotes the *join*—the smallest element of $\mathcal{L}$ greater than both τ and φ (resp. the *meet*—the largest element of $\mathcal{L}$ smaller than both τ and φ)). These varieties were extensively studied by Hibi in [10], where it is proved that $X(\mathcal{L})$ is a normal variety. On the other hand, Eisenbud and Sturmfels [6] showed that a binomial prime ideal is toric (here, "toric ideal" is in the sense of [17]). Thus one obtains that $X(\mathcal{L})$ is a normal toric variety. We shall refer to such an $X(\mathcal{L})$ as a *Hibi toric variety.*

For $\mathcal{L}$ the Bruhat poset of Schubert varieties in a minuscule G/P, it is shown in [8] that $X(\mathcal{L})$ flatly deforms to $\widehat{G/P}$ (the cone over G/P); in other words, there exists a flat family over $\mathbb{A}^1$ with $\widehat{G/P}$ as the generic fiber and $X(\mathcal{L})$ as the special fiber. More generally, for a Schubert variety $X(w)$ in a minuscule G/P, it is shown in [8] that $X(\mathcal{L}_w)$ flatly deforms to $\widehat{X(w)}$, the cone over $X(w)$ (here, $\mathcal{L}_w$ is the Bruhat poset of Schubert subvarieties of $X(w)$). In a subsequent paper [9], the authors studied the singularities of $X(\mathcal{L})$ for $\mathcal{L}$ the Bruhat poset of Schubert varieties in the Grassmannian; they also gave a conjecture (see [9, Sec. 11]; see also Remark 9.1 of this paper) giving a necessary and sufficient condition for a point on $X(\mathcal{L})$ to be smooth and proved the sufficiency part of the conjecture. Subsequently, the necessary part of the conjecture was proved in [2] by Batyrev and colleagues. The toric varieties $X(\mathcal{L})$ for $\mathcal{L}$ the Bruhat poset of Schubert varieties in the Grassmannian play an important role in the area of mirror symmetry; for more details, see [1; 2]. We refer to such an $X(\mathcal{L})$ as a *Grassmann–Hibi toric variety* (or G-H toric variety).

The proof (in [9]) of the sufficiency part of the conjecture in [9] uses the Jacobian criterion for smoothness, whereas the proof (in [2]) of the necessary part of the conjecture in [9] uses certain desingularization of $X(\mathcal{L})$.

It should be remarked that neither [9] nor [2] discusses the relationship between the singularities of $X(\mathcal{L})$ and the combinatorics of the polyhedral cone associated

Received November 10, 2008. Revision received March 25, 2009.
The second author was partially supported by NSF Grant DMS-0652386 and Northeastern University RSDF 07-08.