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Iterates of Vinogradov's Quadric and Prime Paucity

Valentin Blomer & Jörg Brüdern

1. Introduction

Vinogradov's quadric is the variety defined by the pair of equations

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2, \qquad x_1 + x_2 + x_3 = y_1 + y_2 + y_3,$$

which is the special case $k = 2$ of the more general system

$$\begin{aligned} x_{1,1} + x_{2,1} + x_{3,1} &= x_{1,j} + x_{2,j} + x_{3,j} \quad (2 \le j \le k), \\ x_{1,1}^2 + x_{2,1}^2 + x_{3,1}^2 &= x_{1,j}^2 + x_{2,j}^2 + x_{3,j}^2 \quad (2 \le j \le k). \end{aligned} \tag{1}$$

In this paper, we study the distribution of integral solutions to (1). Our first result concerns the number $V_k(N)$ of such solutions inside the sphere

$$x_{1,j}^2 + x_{2,j}^2 + x_{3,j}^2 \le 3N^2, \quad 1 \le j \le k. \tag{2}$$

Theorem 1. *Let $k \ge 2$ be a natural number. Then, for any real number δ with $0 < \delta < 3/2^k$,*

$$V_k(N) = N^3 P_k(\log N) + O(N^{3-\delta}), \tag{3}$$

where P_k is a polynomial of degree $2^{k-1} - 1$. In particular, $P_2(x) = 48(x + c)$, where

$$c = \gamma + \frac{1}{2}\log 2 + \log 3 - \frac{4}{3} + \frac{L'(1,\chi)}{L(1,\chi)} - \frac{\zeta'(2)}{\zeta(2)}$$

and where χ is the nontrivial character modulo 3. Moreover,

$$V_2(N) = N^3 P_2(\log N) + O(N^2 \log N). \tag{4}$$

The error term in (3) stems from the use of Weyl's bound for $\zeta(s)$ and $L(s,\chi)$ in the critical strip, and it can be improved by working with a truncated version of the Mellin integral (see equation (21) in Section 3) and with better bounds for $\zeta(s)$ and $L(s,\chi)$. When $k = 2$, one can use fourth moments of these functions over the critical line to obtain (4). If the Lindelöf hypothesis were true for $\zeta(s)$ and $L(s,\chi)$, then for any $k \ge 3$ the formula (3) would hold for any $\delta < 1$.

From the point of view of arithmetic geometry it is perhaps more natural to count solutions of (1) inside the box $|x_{i,j}| \le N$. Let $\tilde{V}_k(N)$ denote the number

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