Michigan Math. J. 59 (2010), 211–230

On a Construction of L. Hua for Positive Reproducing Kernels

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1. Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain (i.e., a connected open set). Following the general rubric of "Hilbert space with reproducing kernel" laid down by N. Aronszajn [Aro], both the Bergman space $A^2(\Omega)$ and the Hardy space $H^2(\Omega)$ have reproducing kernels. We shall provide the details of these assertions below.

The Bergman kernel (for A^2) and the Szegő kernel (for H^2) both have the advantage of being canonical. But neither is positive, and this makes them tricky to handle. The Bergman kernel can be treated with the theory of the Hilbert integral (see [PSt]), and the Szegő kernel can often be handled with a suitable theory of singular integrals (see [K7]).

It is a classical construction of Hua [H] that one can use the Szegő kernel to produce another reproducing kernel $\mathcal{P}(z,\zeta)$ that also reproduces H^2 but is positive; in this sense, it is more like the Poisson kernel of harmonic function theory. In fact, this so-called Poisson–Szegő kernel coincides with the Poisson kernel when the domain is the disc D in the complex plane $\mathbb{C}$. Furthermore, the Poisson–Szegő kernel solves the Dirichlet problem for the invariant Laplacian (i.e., the Laplace–Beltrami operator for the Bergman metric) on the ball in $\mathbb{C}^n$. Unfortunately, a similar statement about the Poisson–Szegő kernel cannot be made on any other domain (although in this paper we explore substitute results on strongly pseudoconvex domains).

Our aim is to develop these ideas with the Szegő kernel replaced by the Bergman kernel. This notion was developed independently by Berezin [Ber] in the context of quantization of Kähler manifolds. Indeed, one assigns to a bounded function on the manifold the corresponding Toeplitz operator. This process of assigning a linear operator to a function is called *quantization*. A nice exposition of the ideas appears in [Pe]; additional basic properties may be found in [Z].

Approaches to the Berezin transform may be operator-theoretic (see [E1; E2]) or geometric [Pe]. The point of view taken in this paper will be more function-theoretic. We shall repeat (in perhaps new language) some results that are known in other contexts. We shall also enunciate and prove new results; many of the

Received October 14, 2008. Revision received February 5, 2009.
The author was supported in part by a grant from the National Science Foundation and a grant from the Dean of Graduate Studies at Washington University.