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Boundedness for Commutators of Rough Hypersingular Integrals with Variable Kernels

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1. Introduction

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ $(n \ge 2)$ with area element $d\sigma(x')$. A function $\Omega(x,z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be in $L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, $q \ge 1$, if Ω satisfies the following conditions:

(1) for any $x, z \in \mathbb{R}^n$, and $\lambda > 0$, $\Omega(x,\lambda z) = \Omega(x,z)$;

(2) $\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})} := \sup_{x\in\mathbb{R}^n}\big(\int_{S^{n-1}}|\Omega(x,z')|^q\,d\sigma(z')\big)^{1/q} < \infty$, where $z' = z/|z|$ for any $z \in \mathbb{R}^n \setminus \{0\}$.

For $\gamma \ge 0$, we define the operator T_γ with variable kernel by

$$T_\gamma f(x) = \text{p.v.}\int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^{n+\gamma}} f(y)\,dy,$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ and $\Omega \in L^\infty(\mathbb{R}^n) \times L^1(S^{n-1})$ satisfies

$$\int_{S^{n-1}} \Omega(x,z')Y_m(z')\,d\sigma(z') = 0 \quad \text{for any } x \in \mathbb{R}^n \tag{1.1}$$

for all spherical harmonic polynomials Y_m with degree $\le [\gamma]$. In the sequel, we denote $T_0 = T$ when $\gamma = 0$ for simplicity.

Obviously, T is the singular integral operator with variable kernel, which was first studied by Calderón and Zygmund in [2]. They found that these operators connect closely to the problem about the second-order linear elliptic equations with variable coefficients. Calderón and Zygmund obtained the following result.

Theorem A (see [2] or [3]). *If* $\Omega(x,z') \in L^\infty(\mathbb{R}^n)\times L^q(S^{n-1}), q > 2(n-1)/n$, *satisfies*

$$\int_{S^{n-1}} \Omega(x,z')\,d\sigma(z') = 0 \quad \textit{for any } x \in \mathbb{R}^n, \tag{1.2}$$

then there is a constant $C > 0$ *such that* $\|Tf\|_{L^2} \le C\|f\|_{L^2}$.

In [16], for $\gamma > 0$ the operator T_γ is called the hypersingular integral operator with variable kernel. Chen, Fan, and Ying [4] extended Theorem A to the homogeneous

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