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Comodules for Some Simple $\mathcal{O}$-forms of $\mathbb{G}_m$

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Tannakian theory allows one to understand an affine group scheme G over a commutative base ring A in terms of the category $\text{Rep}(G)$ of G-modules, by which is meant comodules for the Hopf algebra corresponding to G. The theory is especially well developed [Sa] in the case that A is a field, and some parts of the theory still work well over more general rings A, say discrete valuation rings (see [Sa; W]).

When A is a field of characteristic 0 and G is connected reductive, the category $\text{Rep}(G)$ is very well understood. However, with the exception of groups as simple as the multiplicative and additive groups, little seems to be known about what $\text{Rep}(G)$ looks like concretely when A is no longer assumed to be a field, even in the most favorable case in which A is a discrete valuation ring and G is a flat affine group scheme over A with connected reductive general fiber.

The modest goal of this paper is to give a concrete description of $\text{Rep}(G)$ for certain flat group schemes G over a discrete valuation ring $\mathcal{O}$ such that the general fiber of G is $\mathbb{G}_m$. It should be noted that $\mathcal{O}$-forms of $\mathbb{G}_m$ are natural first examples to consider, as $\mathbb{G}_m/\mathbb{Q}_p$ arises in the Tannakian description [Sa] of the category of isocrystals with integral slopes.

Choose a generator π of the maximal ideal of $\mathcal{O}$ and write F for the field of fractions of $\mathcal{O}$. For any nonnegative integer k, the construction of Section 1.1, when applied to $f = \pi^k$, yields a commutative flat affine group scheme G_k over $\mathcal{O}$ whose general fiber is $\mathbb{G}_m$. The $\mathcal{O}$-points of G_k are given by

$$G_k(\mathcal{O}) = \{t \in \mathcal{O}^\times : t \equiv 1 \bmod \pi^k\},$$

a principal congruence subgroup arising naturally in the much more general context of Moy–Prasad [MoP] subgroups of p-adic reductive groups. These form a projective system

$$\cdots \to G_2 \to G_1 \to G_0 = \mathbb{G}_m$$

in an obvious way, and we may form the projective limit $G_\infty := \text{proj lim}\, G_k$. The Hopf algebra S_k corresponding to G_k can be described explicitly (see Sections 1.1 and 1.2). The Hopf algebra S_∞ corresponding to G_∞ is

$$\text{inj lim}\, S_k = \left\{ \sum_{i\in\mathbb{Z}} x_i T^i \in F[T, T^{-1}] : \sum_{i\in\mathbb{Z}} x_i \in \mathcal{O} \right\}.$$

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