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Generically Ordinary Fibrations and a Counterexample to Parshin's Conjecture

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1. Introduction

For a proper smooth surface X of general type over the field of complex numbers $\mathbb{C}$, the Miyaoka–Yau inequality states a relation between two Chern numbers of X:

$$c_1^2(X) \le 3c_2(X).$$

However, the Miyaoka–Yau inequality does not hold in general over a field of positive characteristic. For example, let us consider $\pi: X \to C$, a generically smooth nonisotrivial semistable fibration of a proper smooth surface to a proper smooth curve over a field of positive characteristic. If both the base genus and the fiber genus are greater than 1, then X is a minimal surface of general type. Let $\pi^{(p^n)}: X^{(p^n)} \to C$ be the base change of π by the n-iterative Frobenius morphism $F^n: C \to C$, and let $\tilde{X}^{(p^n)} \to X^{(p^n)}$ be the minimal desingulization of $X^{(p^n)}$. Then it can be easily checked that, for any $M > 0$, if n is sufficiently large then $\tilde{X}^{(p^n)}$ violates the inequality $c_1^2 \le Mc_2$ [14, p. 195]. On the other hand, in a letter to D. Zagier, Parshin [13, p. 288] proposed that a version of the Miyaoka–Yau inequality might hold for a surface of general type whose Picard scheme is smooth. In this paper, we will construct a counterexample to this conjecture.

Theorem. *For any $M > 0$, there is a smooth proper surface of general type X over a finite field whose Picard scheme is smooth and $c_1^2(X) > Mc_2(X)$.*

The key step in the construction is the following observation.

Lemma 2.10. *If $\pi: X \to C$ is a generically ordinary semistable fibration, then*

$$\dim H^0(R^1\pi_*\mathcal{O}_X) = \dim H^0(R^1\pi_*^{(p^n)}\mathcal{O}_{X^{p^n}})$$

and

$$\dim H^1(\mathcal{O}_X) = \dim H^1(\mathcal{O}_{X^{p^n}})$$

for any n.

From these facts and the Rieman–Roch theorem, we easily obtain the following result.

Corollary 2.11. *Under the same condition as in Lemma 2.10, all the Harder–Narasimhan slopes of $R^1\pi_*(\mathcal{O}_X)$ are nonpositive.*

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