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Wiener's Positive Fourier Coefficients Theorem in Variants of L^p Spaces

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1. Introduction

In this paper we consider spaces that are "close" to $L^p(\mathbb{T})$: L^p itself; the space of functions f with positive Fourier coefficients that have $|f|^p$ integrable near 0; the space of functions whose Fourier coefficients are in $\ell^{p'}$; the space of functions whose Fourier coefficients $\{c_n\}$ satisfy $\sum |c_n|^p n^{p-2} < \infty$; and the mixed norm spaces $\ell^{p',2}$, $1 < p < 2$. We shall describe several relationships between these spaces.

Let $\mathbb{T}$ be the interval $[-\pi,\pi]$. For every $1 \le p < \infty$, we say that a measurable function f is in $L^p = L^p(\mathbb{T})$ if

$$\|f\|_p^p = \frac{1}{2\pi}\int_{\mathbb{T}} |f(x)|^p \, dx < \infty.$$

Note that $L^p \subseteq L^1$ for every $p \ge 1$. For $f \in L^1$ and for every integer n, let

$$\hat f(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x) e^{-inx} \, dx \tag{1.1}$$

be the nth Fourier coefficient of f and let $\sum \hat f(n) e^{inx}$ be the Fourier series of f. For each $p > 1$, let

$$L^p_{\mathrm{loc}+} = \left\{ f : \text{all } \hat f(n) \ge 0 \text{ and } \int_{-\delta}^{\delta} |f|^p \, dx < \infty \text{ for some } \delta = \delta(f) > 0 \right\}.$$

An unpublished theorem of Norbert Wiener asserts that if $f \in L^2_{\mathrm{loc}+}$ then $f \in L^2(\mathbb{T})$. The short proof involves observing that, for each n, $\hat f(n) \le$ a constant times $|\widehat{hf}(n)|$, where

$$h(x) = \begin{cases} 1 - |x/\delta|, & x \in [-\delta,\delta], \\ 0, & x \in \mathbb{T} \setminus [-\delta,\delta], \end{cases}$$

so that $hf \in L^2(\mathbb{T})$. Thus $\sum |\widehat{hf}(n)|^2 < \infty$, $\sum |\hat f(n)|^2 < \infty$, and $f \in L^2(\mathbb{T})$ by Parseval's theorem. Much later, Stephen Wainger remarked that $f \in L^{2n}_{\mathrm{loc}+}$ implies $f \in L^{2n}_{\mathrm{loc}+}$, $n = 1,2,3,\dots$, but gave examples showing that $f \in L^p_{\mathrm{loc}+}$ does not necessarily imply that $f \in L^p(\mathbb{T})$ when $1 < p < 2$. Next, Harold Shapiro showed that if $p \in (2,\infty)$ is not an even integer then $f \in L^p_{\mathrm{loc}+}$ does not necessarily imply

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