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Maximal Operator for Pseudodifferential Operators with Homogeneous Symbols

Yoshihiro Sawano

1. Introduction

The class S^0 is a basic class of pseudodifferential operators that has been investigated by many authors. For example, it is quite fundamental that the pseudodifferential operators with symbol S^0 are L^2-bounded (see [14]). However, given that $L^2 \simeq \dot{F}^0_{22}$ (where $\dot{F}^0_{22}$ is the homogeneous Triebel–Lizorkin space), it seems there is no need to assume $\sup_{x\in\mathbb{R}^n,\,|\xi|\le 1}|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| < \infty$ for all multiindices α,β. Indeed, Grafakos and Torres established that it suffices to assume

$$c_{\alpha,\beta}(a) := \sup_{x\in\mathbb{R}^n,\,\xi\in\mathbb{R}^n} |\xi|^{|\alpha|-|\beta|}|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| < \infty \tag{1}$$

for all multiindices α,β. Denote by $a(x,D)^\sharp$ the formal adjoint of $a(x,D)$. It is natural to assume that

$$a(x,D)^\sharp 1(x) = 0, \tag{2}$$

since one must postulate some moment condition on atoms for $\dot{F}^0_{22}$ when considering the atomic decomposition (see [2; 15]).

We shall assume that $a \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ is a function satisfying (1) and (2). In [5], Grafakos and Torres established that

$$f \in \mathcal{S}_0 \mapsto \int_{\mathbb{R}^n} a(x,\xi)\exp(2\pi i x\cdot\xi)\mathcal{F}^{-1}f(\xi)\,d\xi$$

extends to an L^2-bounded operator, where $\mathcal{S}_0$ denotes the closed subspace of $\mathcal{S}$ that consists of the functions with vanishing moment of any order.

We seek to obtain a maximal estimate related to this operator. To formulate our results, we need some notation. Given $a,\xi \in \mathbb{R}^n$ and $\lambda > 0$, define

$$\begin{aligned} T_a f(x) &:= f(x-a),\\ M_\xi f(x) &:= \exp(2\pi i \xi\cdot x) f(x),\\ D_\lambda f(x) &:= \lambda^{-n/2} f(\lambda^{-1}x). \end{aligned}$$

We use $A \lesssim_{X,Y,\dots} B$ to denote that there exists a constant $c > 0$, depending only on the parameters $X, Y, \dots$, such that $A \le cB$. If the constant c depends only on

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