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On the Topology of Surface Singularities $\{z^n = f(x, y)\}$ for f Irreducible

Elizabeth A. Sell

1. Introduction

Let $(X, 0) \subset (\mathbb{C}^k, 0)$ be the germ of a complex analytic normal surface singularity. The intersection of X with a sufficiently small sphere centered at the origin in $\mathbb{C}^k$ is a compact connected oriented 3-manifold Σ, called the *link* of $(X, 0)$, that does not depend upon the embedding in $\mathbb{C}^k$. Let Γ be the dual resolution graph of a good resolution of the singularity. The homeomorphism type of the link can be recovered from Γ; conversely, Neumann [8] proved that (aside from a few exceptions) the homeomorphism type of the link determines the minimal good resolution graph. One interesting class of normal surface singularities is the set of those for which the link is a *rational homology sphere* ($\mathbb{Q}$HS) (i.e., $H_1(\Sigma, \mathbb{Q}) = 0$). The link is a $\mathbb{Q}$HS if and only if any good resolution graph Γ of $(X, 0)$ is a tree of rational curves.

The work of Neumann and Wahl (described in Section 2; see also [10; 18]) provides a method for generating analytic data for singularities from topological data. Starting with a resolution graph Γ that satisfies certain conditions, known as the "semigroup and congruence conditions", one can produce defining equations for a normal surface singularity with resolution graph Γ. The singularities that result from this algorithm are called *splice quotients.* If the link Σ is a $\mathbb{Z}$HS ($H_1(\Sigma, \mathbb{Z}) = 0$), then only the semigroup conditions are relevant, and the singularities produced by the algorithm are said to be *of splice type*. This work has led to a recent interest in the properties of splice quotients and related topics (see [3; 6; 13; 14; 17]), and there are still many unanswered questions.

One of the first questions that arises is: How many singularities with $\mathbb{Q}$HS link are splice quotients? There are two layers to the problem, topological and analytic. If one has a singularity that satisfies the necessary topological conditions (which depend only on the resolution graph), then there exist splice quotients with that topological type—but it is a separate issue to determine whether the singularity is analytically isomorphic to a splice quotient. Originally, one wondered whether all $\mathbb{Q}$-Gorenstein singularities with $\mathbb{Q}$HS link would turn out to be splice quotients. However, the first counterexamples were found in the paper of Luengo-Velasco, Melle-Hernández, and Némethi [3]. The authors give an example of a hypersurface singularity for which the resolution graph does not satisfy the semigroup

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