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Smoothings of Schemes with Nonisolated Singularities

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1. Introduction

The purpose of this paper is to describe the deformation and $\mathbb{Q}$-Gorenstein deformation theory of schemes defined over a field k with nonisolated singularities and to obtain criteria for the existence of smoothings and $\mathbb{Q}$-Gorenstein smoothings. The motivation for doing so comes from many different problems. Two of the most important ones are the compactification of the moduli space of surfaces of general type (and its higher-dimensional analogues) and the minimal model program.

Let $0 \in C$ be the germ of a smooth curve and let $U = C - 0$. It is well known [A; KoSh] that any family $f_U : \mathcal{X}_U \to U$ of smooth surfaces of general type over U can be completed in a unique way to a family $f : \mathcal{X} \to C$ such that $\omega^{[k]}_{\mathcal{X}/C}$ is invertible and ample for some $k > 0$ and the central fiber $X = f^{-1}(0)$ is a stable surface. A *stable* surface is a proper 2-dimensional reduced scheme X such that X has only semi-log-canonical singularities and $\omega_X^{[k]}$ is locally free and ample for some $k > 0$. Hence the moduli space of surfaces of general type can be compactified by adding the stable surfaces. Therefore, we should like to know which stable surfaces are smoothable and which are not. For an overview of recent advances in this area and the higher-dimensional analogues, see [A].

We would like to mention two applications from the minimal model program that are related to the smoothability problem.

1. The outcome of the minimal model program starting with a smooth, n-dimensional projective variety X is a terminal projective variety Y such that either K_Y is nef or Y has a Mori fiber space structure, which means that there is a projective morphism $f : Y \to Z$ with $-K_Y$ f-ample. Suppose that the second case occurs and $\dim Z = 1$. Let $z \in Z$ and $Y_z = f^{-1}(z)$. Then Y_z is a Fano variety of dimension $n-1$ and Y is a $\mathbb{Q}$-Gorenstein smoothing Y_z. In general, Y_z has nonisolated singularities and may not even be normal. Hence the classification of Mori fiber spaces in dimension n is directly related to the classification of smoothable Fano varieties of dimension $n - 1$.

2. One of the two fundamental maps that appear in the context of the 3-dimensional minimal model program is an extremal neighborhood. A 3-fold *terminal extremal* neighborhood [KoMo] is a proper birational map $\Delta \subset Y \xrightarrow{f} X \ni P$ such that Y is the germ of a 3-fold along a proper curve Δ, $\Delta_{\text{red}} = f^{-1}(P)$, Y is terminal,

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