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Topological Aspects of Poset Spaces

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1. Introduction

Recent work in mathematical logic [10; 11; 12] has led to an interest in certain topological spaces formed from filters on partially ordered sets. This paper describes the general topology of these poset spaces.

The results of the paper are divided as follows. In Section 2 we define two classes of spaces, MF spaces and UF spaces. Together these spaces form the class of poset spaces. We show that many familiar spaces are homeomorphic to poset spaces. In Section 3, we characterize the separation properties of poset spaces and show that any second-countable poset space is homeomorphic to a space of the same kind formed from a countable poset. In Section 4, we show that the class of MF spaces are closed under arbitrary topological products and that any G_δ subspace of an MF space is again an MF space. We show that UF spaces are closed under the action of taking G_δ subspaces but not closed under binary products. In Section 5, we establish that poset spaces are of the second Baire category and possess the strong Choquet property. We give a characterization of the class of countably based MF spaces as the class of second-countable T_1 spaces with the strong Choquet property. In Section 6, we apply the results of Section 5 to domain theory, giving a complete characterization of the second-countable topological spaces that have a domain representation. Section 7 contains results on the relationship between MF spaces (not necessarily countably based) and semi-topogenous orders. We use semi-topogenous orders to establish a sufficient condition for an arbitrary space to be homeomorphic to an MF space. In Section 8, we show that every second-countable poset space either is countable or contains a perfect closed set.

2. Poset Spaces

Our goal in this section is to define the class of poset spaces and show that this class includes all complete metric spaces and all locally compact Hausdorff spaces. We first review some basic definitions about partially ordered sets.

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