On Fano Manifolds with a Birational Contraction
Sending a Divisor to a Curve

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1. Introduction

Let $X$ be a smooth, complex Fano variety of dimension $n$. The Picard number $\rho_X$ of $X$ is equal to the second Betti number of $X$, and is bounded in any fixed dimension, because $X$ can vary only in a finite number of families (see [De, Chap. 5] and references therein). If $n = 3$ and $\rho_X \geq 6$, then $X \cong S \times \mathbb{P}^1$ where $S$ is a Del Pezzo surface, so that $\rho_X \leq 10$ [MoMu, Thm. 2]. Starting from dimension 4, the maximal value of $\rho_X$ is unknown.

Let’s assume that $n \geq 4$. Bounds on the Picard number are known when $X$ has some special extremal contraction. For instance, if $X$ has a birational elementary contraction sending a divisor to a point, then $\rho_X \leq 3$ ([T2, Prop. 5]; see also Proposition 3.1). In fact such $X$ are classified in the toric case [Bo], in the case of a blow-up of a point [BoCamW], and more generally when the exceptional divisor is $\mathbb{P}^{n-1}$ [T2]. Concerning the fiber type case, we know that $\rho_X \leq 11$ when $X$ has an elementary contraction onto a surface or a 3-fold [Ca2, Thm. 1.1].

In this paper we consider the case of a birational elementary contraction of type $(n-1, 1)$—that is, sending a divisor to a curve. Such Fano varieties have been classified in the toric case by Sato [S], and Tsukioka [T1; T3] has obtained classification results for some cases (see Remark 4.3). Our main result is the following.

**Theorem 1.1.** Let $X$ be a smooth Fano variety of dimension $n \geq 4$, and suppose that $X$ has a birational elementary contraction sending a divisor $E$ to a curve.

Then $\rho_X \leq 5$. Moreover, if $\rho_X = 5$ then we have $E \cong W \times \mathbb{P}^1$ for $W$ a smooth Fano variety, and there exist:

- a smooth projective variety $Y$, with $\rho_Y = 4$, such that $X$ is the blow-up of $Y$ in a subvariety isomorphic to $W$ with exceptional divisor $E$; and
- a smooth Fano variety $Z$, with $\rho_Z = 3$, having a birational elementary contraction sending a divisor $E_Z$ to a curve and such that $X$ is the blow-up of $Z$ in two fibers of this contraction and $E$ is the proper transform of $E_Z$.

This theorem follows from Theorem 4.2 and Proposition 4.8. There are examples with $\rho_X = 5$ in every dimension $n \geq 4$; see Example 4.10. In dimension 4, we get the following.