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On Fano Manifolds with a Birational Contraction Sending a Divisor to a Curve

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1. Introduction

Let X be a smooth, complex Fano variety of dimension n. The Picard number ρ_X of X is equal to the second Betti number of X, and is bounded in any fixed dimension, because X can vary only in a finite number of families (see [De, Chap. 5] and references therein). If $n = 3$ and $\rho_X \geq 6$, then $X \cong S \times \mathbb{P}^1$ where S is a Del Pezzo surface, so that $\rho_X \leq 10$ [MoMu, Thm. 2]. Starting from dimension 4, the maximal value of ρ_X is unknown.

Let's assume that $n \geq 4$. Bounds on the Picard number are known when X has some special extremal contraction. For instance, if X has a birational elementary contraction sending a divisor to a point, then $\rho_X \leq 3$ ([T2, Prop. 5]; see also Proposition 3.1). In fact such X are classified in the toric case [Bo], in the case of a blow-up of a point [BoCamW], and more generally when the exceptional divisor is $\mathbb{P}^{n-1}$ [T2]. Concerning the fiber type case, we know that $\rho_X \leq 11$ when X has an elementary contraction onto a surface or a 3-fold [Ca2, Thm. 1.1].

In this paper we consider the case of a birational elementary contraction of type $(n-1, 1)$—that is, sending a divisor to a curve. Such Fano varieties have been classified in the toric case by Sato [S], and Tsukioka [T1; T3] has obtained classification results for some cases (see Remark 4.3). Our main result is the following.

Theorem 1.1. *Let X be a smooth Fano variety of dimension $n \geq 4$, and suppose that X has a birational elementary contraction sending a divisor E to a curve.*

Then $\rho_X \leq 5$. Moreover, if $\rho_X = 5$ then we have $E \cong W \times \mathbb{P}^1$ for W a smooth Fano variety, and there exist:

- *a smooth projective variety Y, with $\rho_Y = 4$, such that X is the blow-up of Y in a subvariety isomorphic to W with exceptional divisor E; and*
- *a smooth Fano variety Z, with $\rho_Z = 3$, having a birational elementary contraction sending a divisor E_Z to a curve and such that X is the blow-up of Z in two fibers of this contraction and E is the proper transform of E_Z.*

This theorem follows from Theorem 4.2 and Proposition 4.8. There are examples with $\rho_X = 5$ in every dimension $n \geq 4$; see Example 4.10. In dimension 4, we get the following.

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