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Borel–Moore Homology and K-theory on the Steinberg Variety

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1. Introduction

Let G be a simply connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$, $\mathcal{B}$ the flag variety of G, $\mathcal{N}$ the nilpotent cone in $\mathfrak{g}$, and $\widetilde{\mathcal{N}}$ the Springer resolution $\{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\}$ of $\mathcal{N}$. We also let

$$Z = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}} = \{(x, \mathfrak{b}, \mathfrak{b}') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{b} \cap \mathfrak{b}'\}$$

be the Steinberg variety. Since its inception, the Steinberg variety has proved to be an important object in the representation theory of Weyl groups and affine Hecke algebras. In particular, if we let $H_*(Z, \mathbb{C}) = \bigoplus_{i \geq 0} H_i(Z, \mathbb{C})$ be the complex graded Borel–Moore homology algebra equipped with a convolution product, then the knowledge of the graded algebra structure of $H_*(Z, \mathbb{C})$ is a key ingredient for obtaining all irreducible representations of Weyl groups through the decomposition theorem of Bellinson, Bernstein, and Deligne.

In this paper, we study more explicitly the $\mathbb{C}$-algebra structure of $H_*(Z, \mathbb{C})$. To do so, we will examine the convolution product of $H_*(Z, \mathbb{C})$ through multiplications on other objects, such as the Grothendieck group of a graph variety, (co)homology of the flag variety, and a certain crossed product algebra. We prove that the convolution product on $H_*(Z, \mathbb{C})$ is compatible with all of these multiplications (see Theorem 3.7). As an application of our compatibility theorem, we construct a $\mathbb{C}$-algebra isomorphism between $H_*(Z, \mathbb{C})$ and a certain crossed product algebra (see Theorem 3.9).

After this paper was written, the author was informed that Douglass and Röhrle [DR1; DR2] proved a result similar to Corollary 3.10. However, our approach differs from that of Douglass and Röhrle.

2. Preliminaries

Borel–Moore Homology. Let X be a complex algebraic variety, and let $\hat{X} = X \cup \{\infty\}$ be the one-point compactification of X. Then the ith Borel–Moore homology space is defined as $H_i(X) := H_i^{\text{ord}}(\hat{X}, \infty)$, where H_i^{ord} denotes the ith singular relative homology over the complex coefficients (see [BoMo] for details). Throughout the paper, we will consider all Borel–Moore homology spaces to be over the complex numbers $\mathbb{C}$.

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