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Isometric Rigidity in Codimension 2

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1. Introduction

In the local theory of submanifolds, a fundamental but difficult problem is to describe the isometrically deformable isometric immersions $f: M^n \to \mathbb{R}^{n+p}$ into Euclidean space with low codimension p as compared with the dimension $n \geq 3$ of the Riemannian manifold. Moreover, one would like to understand the set of all possible isometric deformations.

Submanifolds in low codimension are generically rigid because the fundamental Gauss–Codazzi–Ricci system of equations is overdetermined. By "rigid" we mean that there are no other isometric immersions up to rigid motion of the ambient space. As a consequence, it is much easier to employ a generic assumption that implies rigidity than to describe the submanifolds that are isometrically deformable. For instance, the results in [1] and [5] establish rigidity *provided* the second fundamental is sufficiently "complicated".

The result stated by Beez [3] in 1876, but not correctly proved (by Killing [13]) until 1885, states that any deformable hypersurfaces without flat points has two nonzero principal curvatures (rank 2) at any point. For dimension 3, the deformation problem for hypersurfaces was first considered by Schur [16] as early as 1886 and then completely solved in 1905 by Bianchi [4]. The general case was solved by Sbrana [15] in 1909 and by Cartan [6] in 1916; see [10] for additional information. From their results, we have that even hypersurfaces of rank 2 are generically rigid.

Outside the hypersurfaces case, the deformation question remains essentially unanswered to this day even for low codimension $p = 2$. According to [5] or [9], any submanifold $f: M^n \to \mathbb{R}^{n+2}$ is rigid if (a) at any point the index of relative nullity satisfies $\nu_f \geq n - 5$ and (b) any shape operator has at least three nonzero principal curvatures. If only the relative nullity condition holds then we know from [12] that f is genuinely rigid. This means that, given any other isometric immersion $\tilde{f}: M^n \to \mathbb{R}^{n+2}$, there is an open dense subset of M^n such that, when restricted to any connected component, either $f|_U$ and $\hat{f}|_U$ are congruent or there exist an isometric embedding $j: U \hookrightarrow N^{n+1}$ into a Riemannian manifold N^{n+1} and either flat or isometric Sbrana–Cartan hypersurfaces $F, \hat{F}: N^{n+1} \to \mathbb{R}^{n+2}$ such that $f|_U = F \circ j$ and $\hat{f}|_U = \hat{F} \circ j$.

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