Michigan Math. J. 58 (2009)

Some Results on the Second Gaussian Map for Curves

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1. Introduction

The first Gaussian map for the canonical series has been intensively studied. It has been shown that, for a general curve of genus different from 9 and ≤ 10, the first Gaussian map is injective, while for genus ≥ 10 and different from 11 it is surjective [7; 9; 20]. In [21] it is proved that if a curve lies on a $K3$ surface then the first Gaussian map cannot be surjective, and it is known (see [18]) that the general curve of genus 11 lies on a $K3$ surface.

In this paper we study some properties of the second Gaussian map

$$\mu_2 \colon I_2(K_X) \to H^0(X, 4K_X).$$

Our geometrical motivation comes from its relation with the curvature of the moduli space M_g of curves of genus g endowed with the Siegel metric induced by the period map $j \colon M_g \to A_g$, which we started to analyze in [10]. There the curvature is computed using the formula for the associated second fundamental form given in [11]. In particular, in [11] it is proved that the second fundamental form lifts the second Gaussian map μ_2, as stated in an unpublished paper of Green and Griffiths (cf. [15]).

In [10, Cor. (3.8)] we give a formula for the holomorphic sectional curvature of M_g along the a Schiffer variation ξ_P, for P a point on the curve X, in terms of the holomorphic sectional curvature of A_g and the second Gaussian map.

The relation of the second Gaussian map with curvature properties of M_g in A_g suggests that its rank could give information on the geometry of M_g. Note that surjectivity can be expected for a general curve of genus at least 18. Recall that M_g is uniruled for $g \leq 15$, has Kodaira dimension at least 2 for $g = 23$, and is of general type for all other values of $g \geq 22$.

Along these lines, in this paper we exhibit infinitely many examples of curves lying on the product of two curves with surjective second Gaussian map. Other examples of curves whose second Gaussian map is surjective were given in [4] for complete intersections. Both classes of examples generalize constructions given by Wahl [21; 22] for the first Gaussian map.

Received May 27, 2008. Revision received July 23, 2008.

The present research took place in the framework of the PRIN 2005 of MIUR, “Spazi dei moduli e teoria di Lie”, and PRIN 2006 of MIUR, “Geometry on algebraic varieties”.