Michigan Math. J. 58 (2009)

Positivity of Cotangent Bundles

Kelly Jabbusch

1. Introduction

Let X be a projective scheme over an algebraically closed field. Given a vector bundle $\mathscr{E}$ on X, we can consider various notions of positivity for $\mathscr{E}$, such as ample, nef, and big. As a particular example, consider a smooth projective variety X and its cotangent bundle Ω_X. When Ω_X is ample, X has some very nice properties. For example, all subvarieties of X are of general type and X is algebraically hyperbolic; so, in particular, X does not contain rational or elliptic curves and there do not exist nonconstant maps $f: A \to X$ where A is an abelian variety and X is Kobayashi hyperbolic [3]. Requiring that the cotangent bundle be ample is certainly a very strong property, and for a long time there were few examples of such varieties even though they were expected to be reasonably abundant. One such example was constructed by Michael Schneider.

Theorem 1.1 [17]. *Let $f: X \to Y$ be a smooth projective nonisotrivial morphism, where X and Y are smooth projective varieties over $\mathbb{C}$ of dimensions 2 and 1, respectively. Suppose that, for all $y \in Y$, the Kodaira–Spencer map $\rho_{f,y}: T_{Y,y} \to H^1(X_y, T_{X_y})$ is nonzero. Then Ω_X is ample.*

Note that certain Kodaira surfaces satisfy the stipulated conditions. In this paper, we generalize Theorem 1.1 to varieties of higher dimensions. To do so, we will introduce a slightly weaker notion of ampleness, which we call "quasi-ample" and "quasi-ample with respect to an open subset U" (see Definitions 1.9 and 1.13). Using this notion, we extend Schneider's result to varieties of higher dimension.

Theorem 1.2. *Let*

$$X^n \xrightarrow{f_n} X^{n-1} \xrightarrow{f_{n-1}} X^{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_3} X^2 \xrightarrow{f_2} X^1,$$

where each X^i is a smooth projective variety over $\mathbb{C}$ of dimension i and each $f_i: X^i \to X^{i-1}$ is a smooth projective morphism with $\operatorname{Var}(f_i) = i-1$. Then Ω_{X^n} is nef and quasi-ample with respect to an open U_n (as defined precisely in Theorem 2.11), and $\mathscr{O}_{\mathbb{P}(\Omega_{X^n})}(1)$ is a big line bundle on $\mathbb{P}(\Omega_{X^n})$.

We also extend this result to towers of varieties

Received May 19, 2008. Revision received July 14, 2008.