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Universal Functions with Prescribed Zeros and Interpolation Properties

Luis Bernal-González, Antonio Bonilla, & Markus Nieß

Dedicated to Professor José Méndez on the occasion of his sixtieth birthday

1. Introduction

Roughly speaking, universality means "existence of a dense orbit". Thus, in some sense, universal functions are "uncontrolled". In this paper, we study the existence of functions that are universal with respect to differential operators and that are, at the same time, "controlled" by prescribed interpolation properties, including prescribed zeros and multiplicities. Precise definitions are given in what follows.

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$, and $\mathbb{N}_0$ the set of positive integers, the set of all integers, the complex plane, and the set $\mathbb{N} \cup \{0\}$, respectively. If $A \subset \mathbb{C}$ then A°, $\bar{A}$, and ∂A will stand (respectively) for the interior, the closure, and the boundary of A in $\mathbb{C}$. We use $\mathbb{C}_\infty$ to denote the extended complex plane. Recall that a *domain* is a nonempty connected open subset of $\mathbb{C}$.

Let $H(\Omega)$ be the linear space of holomorphic functions on a domain Ω. In particular, $H(\mathbb{C})$ is the space of entire functions. Consider the metric

$$d(f,h) := \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{\|f-h\|_{C_j}}{1+\|f-h\|_{C_j}} \quad (f,h \in H(\Omega)),$$

where

$$\|f-h\|_M := \sup_{z\in M} |f(z)-h(z)|.$$

Here $\{C_j : j \ge 1\}$ is a fixed exhaustive sequence of compact subsets of Ω; that is, $C_j \subset C_{j+1}^\circ$ $(j \ge 1)$ and $\Omega = \bigcup_{j=1}^\infty C_j$. It is possible to select $\{C_j : j \ge 1\}$ so that each connected component of $\mathbb{C}_\infty \setminus C_j$ contains some connected component of $\mathbb{C}_\infty \setminus \Omega$; in particular, if Ω is simply connected (i.e., if $\mathbb{C}_\infty \setminus \Omega$ is connected) then we can choose every C_j without "holes".

The aforementioned metric d generates on $H(\Omega)$ the topology of uniform convergence on compact subsets of Ω; see [5]. In the sequel, we will always consider the complete metric space $(H(\Omega), d)$.

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