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Wonderful Compactification of an Arrangement of Subvarieties

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1. Introduction

The purpose of this paper is to define the so-called wonderful compactification of an arrangement of subvarieties, to prove its expected properties, to give a construction by a sequence of blow-ups, and to discuss the order in which the blow-ups can be carried out.

Fix a nonsingular algebraic variety Y over an algebraically closed field (of arbitrary characteristic). An *arrangement* of subvarieties $\mathcal{S}$ is a finite collection of nonsingular subvarieties such that all nonempty scheme-theoretic intersections of subvarieties in $\mathcal{S}$ are again in $\mathcal{S}$ or, equivalently, such that any two subvarieties intersect cleanly and the intersection is either empty or a subvariety in this collection (see Definition 2.1).

Let $\mathcal{S}$ be an arrangement of subvarieties of Y. A subset $\mathcal{G} \subseteq \mathcal{S}$ is called a *building set of* $\mathcal{S}$ if, for all $S \in \mathcal{S} \setminus \mathcal{G}$, the minimal elements in $\{G \in \mathcal{G} : G \supseteq S\}$ intersect transversally and the intersection is S. A set of subvarieties $\mathcal{G}$ is called a building set if all the possible intersections of subvarieties in $\mathcal{G}$ form an arrangement $\mathcal{S}$ (called the *induced arrangement* of $\mathcal{G}$) and $\mathcal{G}$ is a building set of $\mathcal{S}$ (see Definition 2.2).

For any building set $\mathcal{G}$, the wonderful compactification of $\mathcal{G}$ is defined as follows.

Definition 1.1. Let $\mathcal{G}$ be a nonempty building set and $Y^\circ = Y \setminus \bigcup_{G \in \mathcal{G}} G$. The closure of the image of the natural locally closed embedding

$$Y^\circ \hookrightarrow \prod_{G \in \mathcal{G}} Bl_G Y$$

is called the *wonderful compactification* of the arrangement $\mathcal{G}$ and is denoted by $Y_{\mathcal{G}}$.

The following description of $Y_{\mathcal{G}}$ is the main theorem and is proved at the end of Section 2.3. A *$\mathcal{G}$-nest* is a subset of the building set $\mathcal{G}$ satisfying some inductive condition (see Definition 2.3).

Theorem 1.2. *Let Y be a nonsingular variety and let $\mathcal{G}$ be a nonempty building set of subvarieties of Y. Then the wonderful compactification $Y_{\mathcal{G}}$ is a nonsingular variety. Moreover, for each $G \in \mathcal{G}$ there is a nonsingular divisor $D_G \subset Y_{\mathcal{G}}$ such that*:

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