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Notes on π_1 of Smooth Loci of log del Pezzo Surfaces

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1. Introduction

A projective surface R over $\mathbb{C}$ is called a *log del Pezzo surface* if it contains only quotient singularities and if the canonical divisor K_R is an anti-ample $\mathbb{Q}$-divisor. Although the fundamental group of R is always trivial, the fundamental group of the smooth locus $\pi_1(R^{\mathrm{sm}})$ is, in general, not zero. Nevertheless, it is known that such a group is always finite (cf. [GZ; KMc]). The aim of this paper is to determine these groups.

Our approach to this problem is as follows. Given a log del Pezzo surface R, we take the universal cover of its smooth locus R^{sm}. Given that $\pi_1(R^{\mathrm{sm}})$ is finite [GZ; KMc], the Riemann existence theorem (see [SGA1]) states that the universal cover is actually an algebraic variety. Therefore, we can take the normal closure S of R in the function field of this covering space. In this way we obtain a pair $(S, \pi_1(R^{\mathrm{sm}}))$, where S is also a log del Pezzo surface and $\pi_1(R^{\mathrm{sm}})$ is a finite group acting on it, such that for every nontrivial element $g \in \pi_1(R^{\mathrm{sm}})$ the fixed locus S^g is isolated. We can also equivariantly resolve S to get a smooth rational surface carrying the same finite group action. This motivates the following definitions.

1.1. Definition. We call a finite group G acting on a normal surface S an *action with isolated fixed points* (IFP) if S has at worst quotient singularities and if, for every nonunit element $g \in G$, the fixed locus S^g consists of finite points. Similarly, we call (S, G) *birational to an action with IFP* if there is a G-equivariant birational proper model S' of S such that (S', G) is an action with IFP.

Now we can divide our question into three parts:

(1) finding all the birational classes (S, G) that contain a representative $(\tilde{S}, G)$ with IFP;
(2) determining those groups G for which we can choose $(\tilde{S}, G)$ as in (1) with the additional property that $K_{\tilde{S}}$ is anti-ample; and
(3) for any G appearing in part (2), checking for the existence of $(\tilde{S}, G)$ satisfying $\pi_1(\tilde{S}^{\mathrm{sm}}) = e$.

All finite subgroups of the Cremona group are classified in [DI]. Based on their table, we can solve the part (1) of our question.

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