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On the Weak-type Constant of the Beurling–Ahlfors Transform

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1. Introduction

The Beurling–Ahlfors operator B defined on $L^p(\mathbb{C})$ by

$$Bf(z) = \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\,dm(w) \tag{1.1}$$

is a well-known example of a Calderón–Zygmund singular integral operator of convolution type. The operator arises naturally in the study of the regularity of solutions of the Beltrami equation and thus has applications to quasiconformal mapping theory and partial differential equations (see [1; 9; 12; 14; 15]). In particular, the Iwaniec conjecture [12] that the L^p norm

$$\|B\|_p = p^* - 1, \tag{1.2}$$

where $1 < p < \infty$ and $p^* = \max\{p, \frac{p}{p-1}\}$, is partly motivated by its relation to the Gehring–Reich conjecture proved by Astala in [1]. (The lower bound of p^*-1 was obtained by Lehto [16] in 1965.) Recent work has revealed B as an exemplary junction between Fourier analysis and probability, and martingale methods established by Burkholder have led to the present best known estimates on $\|B\|_p$; see [3; 4; 10].

In this paper we investigate the action of B on the radial function subspaces (for m a nonnegative integer)

$$\mathcal{R}_m^p = \{f \in L^p(\mathbb{C}) : f(re^{i\theta}) = H(r)e^{-im\theta}\}, \tag{1.3}$$

and we arrive at a corresponding family of one-dimensional operators $\{\Lambda_m\}_{m\ge 0}$ on $L^p([0,\infty))$ defined by

$$\begin{aligned}\Lambda_m g(u) = (\mathcal{H}_m - I)g(u) &= \frac{2m+2}{m+2}\int_0^1 g(uv^{2/(m+2)})\,dv - g(u)\\ &= \int_0^1 g(uv)(m+1)v^{m/2}\,dv - g(u).\end{aligned} \tag{1.4}$$

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