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Convergence of the Kähler–Ricci Flow and Multiplier Ideal Sheaves on del Pezzo Surfaces

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1. Introduction

Let X be an n-dimensional compact complex manifold with positive first Chern class $c_1(X)$. Such manifolds are called *Fano* manifolds. The Kähler–Ricci flow on X is defined by the equation

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} + g_{i\bar{j}}, \tag{1}$$

where $R_{i\bar{j}} = -\partial_i\partial_{\bar{j}} \log \det g_{\alpha\bar{\beta}}$ is the Ricci curvature tensor of the hermitian metric $\sum_{i,j} g_{i\bar{j}} dz_i \otimes d\bar{z}_j$. If the class of the Kähler form $\hat{\omega} = \frac{i}{2\pi}\sum_{i,j} \hat{g}_{i\bar{j}} dz_i \wedge d\bar{z}_j$ is $c_1(X)$, then the Kähler–Ricci flow preserves the class of $i\sum_{i,j} \hat{g}_{i\bar{j}} dz_i \wedge d\bar{z}_j$, so we can write

$$g_{i\bar{j}} = \hat{g}_{i\bar{j}} + \partial_i\partial_{\bar{j}}\phi$$

for the solution to the Kähler–Ricci flow with initial condition

$$g_{i\bar{j}}(0) = \hat{g}_{i\bar{j}}.$$

Equation (1) can be reformulated as

$$\frac{\partial}{\partial t}\phi = \log \frac{\det g_{\alpha\bar{\beta}}}{\det \hat{g}_{\alpha\bar{\beta}}} + \phi - \hat{f}, \quad \phi(0) = c_0 \in \mathbb{R}, \tag{2}$$

where $\hat{f}$ is the Ricci potential; that is, for $\hat{R}_{i\bar{j}} = -\partial_i\partial_{\bar{j}} \log \det \hat{g}_{\alpha\bar{\beta}}$ we have $\hat{R}_{i\bar{j}} - \hat{g}_{i\bar{j}} = \partial_i\partial_{\bar{j}}\hat{f}$. It was proven in [Ca] that the solution to (1) exists for all $t > 0$. This paper investigates the issue of convergence based on the following theorem, which first appeared in [PSeS]. The version given here, which is stronger than the one in [PSeS], is based on [PS].

Theorem 1.1 [PSeS; PS]. *Let X be a Fano manifold. Consider the Ricci flow in the form of* (2) *with the initial value c_0 specified by* [PSeS, (2.10)]. *Then the following two statements are equivalent.*

(i) *There exists a $p > 1$ such that*

$$\sup_{t\geq 0} \int_X e^{-p\phi}\hat{\omega}^n < \infty.$$

(ii) *The family of metrics $g_{i\bar{j}}(t)$ converges in C^∞-norm exponentially fast to a Kähler–Einstein metric.*

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