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L^2-Betti Numbers of Plane Algebraic Curves

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1. Introduction

Let X be any topological space and let $\varphi\colon \pi_1(X) \to \Gamma$ be a homomorphism to a group (all groups are assumed to be countable). Then for $p \in \mathbb{N} \cup \{0\}$ we can consider the L^2-Betti number $b_p^{(2)}(X, \varphi) \in [0, \infty]$. We recall the definition and some of the most important properties of L^2-Betti numbers in Section 2.

Let $\mathcal{C} \subset \mathbb{C}^2$ be a reduced plane algebraic curve with irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_r$. We write $X(\mathcal{C}) := \mathbb{C}^2 \setminus \nu\mathcal{C}$ for $\nu\mathcal{C}$ a regular neighborhood of $\mathcal{C}$ inside $\mathbb{C}^2$. We denote the meridians about the nonsingular parts of $\mathcal{C}_1, \dots, \mathcal{C}_r$ by $\mu_1, \dots, \mu_r$. Note that these meridians come with a preferred orientation because the nonsingular parts of the irreducible components $\mathcal{C}_i$ are complex submanifolds of $\mathbb{C}^2$.

It is well known (cf. Theorem 3.1) that $H_1(X(\mathcal{C}); \mathbb{Z})$ is the free abelian group generated by the meridians $\mu_1, \dots, \mu_r$. Throughout this paper we denote by ϕ the map $\pi_1(X(\mathcal{C}); \mathbb{Z}) \to \mathbb{Z}$ given by sending each meridian μ_i to 1. We also refer to ϕ as the total linking homomorphism. We henceforth call a homomorphism $\alpha\colon \pi_1(X(\mathcal{C})) \to \Gamma$ to a group *admissible* if the total linking homomorphism ϕ factors through α.

Our first result is the following.

Theorem 1.1. *Let $\mathcal{C} \subset \mathbb{C}^2$ be a reduced algebraic curve $\mathcal{C}$ whose projective completion intersects the line at infinity transversely. Let $\alpha\colon \pi_1(X(\mathcal{C})) \to \Gamma$ be an admissible homomorphism. Then*

$$b_p^{(2)}(X(\mathcal{C}), \alpha) = \begin{cases} 0 & \textit{for } p \neq 2, \\ \chi(X(\mathcal{C})) & \textit{for } p = 2. \end{cases}$$

In [DaJLe] it was shown that if $\mathcal{A}$ is an affine hyperplane arrangement in $\mathbb{C}^n$ then at most one of the L^2-Betti numbers $b_p^{(2)}(\mathbb{C}^n \setminus \mathcal{A}, \mathrm{id})$ is nonzero. Theorem 1.1 can be seen as an analogous statement for the complement of an algebraic curve in $\mathbb{C}^2$ that is in general position at infinity. Note that if Γ is a polytorsion-free abelian (PTFA) group then this theorem, together with Proposition 2.4, recovers [LMa1, Cor. 4.2].

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