Michigan Math. J. 58 (2009)

A Counterexample to Uniform Approximation on Totally Real Manifolds in $\mathbb{C}^3$

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1. Introduction and Main Result

Let X be a closed subset of $\mathbb{C}^n$. We say that X admits *uniform approximation* if for any continuous function $f \in \mathcal{C}(X)$ and for all $\varepsilon > 0$ there exists a holomorphic function $g \in \mathcal{O}(\mathbb{C}^n)$ such that $\sup_{x\in X}|f(x) - g(x)| < \varepsilon$. We will be concerned with the case where X is a totally real manifold: a smooth manifold whose tangent space at no point contains a nontrivial complex subspace (in which case there are no Cauchy–Riemann equations on X!).

One of the first results in approximation theory in complex analysis was the well-known theorem of Weierstrass [15]: If $X \subset \mathbb{C}$ is an in interval of the real line, then X admits uniform approximation. This result was generalized by Carleman [4] to the effect that X could be taken to be the entire real line in the complex plane. A complete characterization of the subsets of $\mathbb{C}$ that admit uniform approximation now exists: X admits uniform approximation if and only if (i) $\mathbb{C} \setminus X$ has no relatively compact components, (ii) X has no interior, and (iii) $\mathbb{C} \setminus X$ is locally connected at infinity (see e.g. [14]). In particular we have that a closed smooth 1-dimensional submanifold of the complex plane admits uniform approximation.

When considering the state of affairs in several complex variables it is natural to consider the compact and noncompact cases separately. Hörmander and Wermer showed that if X is a polynomially convex compact totally real manifold then X admits uniform approximation [7]. It is also possible to get $\mathcal{C}^k$-approximation on X—depending on the smoothness of X and the function f [11].

In the noncompact case the situation is best understood if the "size" of X is small. Following the work of Alexander [1], Gauthier and Zeron [5] have shown that uniform approximation is possible if $X \subset \mathbb{C}^n$ is a locally rectifiable dendrite—that is, if it is closed and connected and has locally finite 1-dimensional measure and if $\check{H}^1(X,\mathbb{Z}) = 0$. In the case of "bigger" sets it is known that one can take $X = \mathbb{R}^n \subset \mathbb{C}^n$ [6; 12]. Manne [9] has shown that if X is the union of two totally real planes such that X is polynomially convex, then X admits uniform approximation.

The purpose of this paper is to produce an example that demonstrates the following theorem.

Theorem 1.1. *There exists a proper $\mathcal{C}^\infty$-smooth embedding $\phi\colon \mathbb{R}^2 \to \mathbb{C}^3$ such that the following statements hold for $M := \phi(\mathbb{R}^2)$:*

Received October 31, 2007. Revision received December 19, 2007.