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Intersection Numbers and Automorphisms of Stable Curves

KEFENG LIU & HAO XU

1. Introduction

Denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n-pointed genus-g complex algebraic curves. We have the morphism that forgets the last marked point

$$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}.$$

Denote by $\sigma_1, \dots, \sigma_n$ the canonical sections of π and by $D_1, \dots, D_n$ the corresponding divisors in $\overline{\mathcal{M}}_{g,n+1}$. Let ω_π be the relative dualizing sheaf. Then we have the following tautological classes on moduli spaces of curves:

$$\psi_i = c_1(\sigma_i^*(\omega_\pi));$$

$$\kappa_i = \pi_*\Big(c_1\Big(\omega_\pi\Big(\sum D_i\Big)\Big)^{i+1}\Big);$$

$$\lambda_l = c_l(\pi_*(\omega_\pi)), \quad 1 \le l \le g.$$

The classes κ_i were first introduced by Mumford [22] on $\overline{\mathcal{M}}_g$; their generalization to $\overline{\mathcal{M}}_{g,n}$ here is due to Arbarello–Cornalba [1]. For background materials about the intersection theory of moduli spaces of curves, we refer to the book [19] and the survey paper [24].

Hodge integrals are intersection numbers of the form

$$\langle \tau_{d_1} \cdots \tau_{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \mid \lambda_1^{k_1} \cdots \lambda_g^{k_g} \rangle := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \lambda_1^{k_1} \cdots \lambda_g^{k_g},$$

which are rational numbers because the moduli spaces of curves are orbifolds. They are nonzero only when $\sum_{i=1}^n d_i + \sum_{i=1}^m a_i + \sum_{i=1}^g i k_i = 3g - 3 + n$.

Hodge integrals arise naturally in the localization computation of Gromov–Witten invariants. They have been extensively studied by mathematicians and physicists. Hodge integrals involving only ψ classes can be computed recursively by the the celebrated Witten–Kontsevich theorem [18; 26], which can be equivalently formulated by the DVV recursion relation [5]

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