Semistandard Filtrations in Highest Weight Categories

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Dedicated to the memory of Donald G. Higman

Introduction

This paper is dedicated to the memory of Donald Higman. Don was especially attracted to methods with some kind of fundamental simplicity but with sufficient substance to be useful and illuminating. This paper is written in the spirit of this lofty aspiration and also somewhat in the direction of Don's early interests in homological algebra.

The paper begins in Section 1 with a foundational discussion of a new notion, that of a semistandard filtration in a highest weight category $\mathcal{C}$. The latter categories [CPS1] axiomatize features found in many Lie-theoretic module settings. For simplicity we stick to the case of a finite weight poset Λ, to which many considerations reduce. Here, the irreducible objects $L(\lambda)$ and projective indecomposable objects $P(\lambda)$ are indexed by Λ, and there are standard objects $\Delta(\lambda)$ with head $L(\lambda)$ and all other composition factors indexed by a smaller weight. The objects $P(\lambda)$ have finite filtrations with sections as standard objects, the top section being $\Delta(\lambda)$ and all others of the form $\Delta(\nu)$ with $\nu > \lambda$. In particular, these conditions imply $\operatorname{Ext}^1_{\mathcal{C}}(\Delta(\lambda), \Delta(\mu)) = 0$ unless $\lambda > \mu$. Thus, whenever an object M in $\mathcal{C}$ has a finite length filtration of subobjects

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = M$$

with sections $F_{i+1}/F_i \simeq \Delta(\nu_i)$ equal to standard objects, we may rearrange the order in which the latter appear to assume that $\nu_i > \nu_j$ never occurs for $i > j$. It is these "standard filtrations" with their additional order requirements (which may always be assumed by rearrangement) that we will generalize to obtain the notion of a semistandard filtration. They will be filtrations in which the role of standard objects is replaced by their nonzero epimorphic images (which might reasonably be called semistandard objects) yet with the order requirements essentially the same. Actually, it will be convenient in the formal definitions to allow direct sums of standard objects as filtration sections for standard objects, and their nonzero epimorphic images in the semistandard case, with similar order requirements. However, these more general filtrations can always be refined to fit the discussion just given for sections that are single standard or semistandard objects.