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Higmanian Rank-5 Association Schemes on 40 Points

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Dedicated to the memory of Donald G. Higman

1. Introduction

We pay our tribute to the mathematical heritage of D. G. Higman in an investigation of imprimitive association schemes on 40 points with four classes that are proper class II in the sense of [H4].

We prove that there are exactly four possible sets of intersection numbers, all of which correspond to a parabolic of type $10 \circ K_4$. There exist exactly 15 association schemes for the first parameter set. The scheme in the case of the second parameter set is unique, up to isomorphism. We provide an example for the third parameter set, but the existence of a scheme for the fourth feasible parameter set remains an open question.

Our results were originally obtained with the aid of a computer, but in many cases we have been able to give computer-free constructions, which are presented here. Additional supporting material is available from ⟨http://www.math.bgu.ac.il/~zivav/math/⟩.

Coherent configurations and coherent algebras are two of the significant concepts introduced by Higman. These concepts are considered in Section 2 in an effort to make our presentation more clear to the reader.

Many of the proofs in this paper are geometric in nature. In a number of cases we even manage to use pictorial arguments, presenting a certain element of a considered group as a visible symmetry of a depicted diagram. In this fashion, following in the spirit of H. S. M. Coxeter, a number of nice auxiliary objects (including the configuration 8_3, the 4-dimensional cube, the Clebsch graph, and the cages on 50 and 40 vertices) are inspected in Section 3.

A short digest of part of Higman's classification of rank-5 imprimitive association schemes is given in Section 4 (a few regrettable typos have been corrected). With the aid of Higman's classification we prove Proposition 4.2, wherein four feasible parameter sets for the schemes on 40 points are enumerated.

In fact, only one such scheme (denoted by $\mathfrak{m}$) was known before. The scheme $\mathfrak{m}$ is generated by the classical Deza graph on 40 vertices. This serves to justify a new axiomatic system for a Deza family in a Higmanian house, as suggested in

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