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Distance-Regular Graphs of q-Racah Type and the q-Tetrahedron Algebra

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In memory of Donald Higman

1. Introduction

In [20], Hartwig and the second author gave a presentation of the three-point $\mathfrak{sl}_2$ loop algebra via generators and relations. To obtain this presentation they defined a Lie algebra $\boxtimes$ by generators and relations and then displayed an isomorphism from $\boxtimes$ to the three-point $\mathfrak{sl}_2$ loop algebra. The algebra $\boxtimes$ is called the *tetrahedron* algebra [20, Def. 1.1]. In [24] we introduced a q-deformation $\boxtimes_q$ of $\boxtimes$ called the q-tetrahedron algebra. In [24] and [25] we described the finite-dimensional irreducible $\boxtimes_q$-modules. In [26, Sec. 4] we displayed four homomorphisms into $\boxtimes_q$ from the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. In [26, Sec. 12] we found a homomorphism from $\boxtimes_q$ into the subconstituent algebra of a distance-regular graph that is self-dual with classical parameters. In this paper we do something similar for a distance-regular graph that is said to have q-Racah type. This type is described as follows. Let Γ denote a distance-regular graph with diameter $D \geq 3$ (See Section 4 for formal definitions). We say that Γ has *q-Racah type* whenever Γ has a Q-polynomial structure with eigenvalue sequence $\{\theta_i\}_{i=0}^D$ and dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^D$ that satisfy, for $0 \leq i \leq D$,

$$\theta_i = \eta + uq^{2i-D} + vq^{D-2i} \quad \text{and}$$
$$\theta^*_i = \eta^* + u^*q^{2i-D} + v^*q^{D-2i},$$

where q, u, v, u^*, v^* are nonzero and $q^{2i} \neq 1$ for $1 \leq i \leq D$. Assume that Γ has q-Racah type.

Fix a vertex x of Γ and let $T = T(x)$ denote the corresponding subconstituent algebra [32, Def. 3.3]. Recall that T is generated by the adjacency matrix A and the dual adjacency matrix $A^* = A^*(x)$ [32, Def. 3.10]. An irreducible T-module W is called *thin* whenever the intersection of W with each eigenspace of A and each eigenspace of A^* has dimension at most 1 [32, Def. 3.5]. Assuming that each irreducible T-module is thin, we display invertible central elements Φ and Ψ of T and a homomorphism $\vartheta : \boxtimes_q \to T$ such that

$$A = \eta I + u\Phi\Psi^{-1}\vartheta(x_{01}) + v\Psi\Phi^{-1}\vartheta(x_{12}) \quad \text{and}$$
$$A^* = \eta^* I + u^*\Phi\Psi\vartheta(x_{23}) + v^*\Psi^{-1}\Phi^{-1}\vartheta(x_{30}),$$

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