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Bounds on Subsets of Coherent Configurations

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Dedicated to D. G. Higman

1. Introduction

Coherent configurations were introduced by D. G. Higman, initially in a 1970 paper [5] and then in a pair of papers [6; 7] that developed the theory. The definition was based on combinatorial properties of the orbitals of a group acting on a finite set, with the intention that the structure would be useful both in group theory and in combinatorics. Some later work (see e.g. [8; 9]) focused on combinatorial aspects, and this paper is in that spirit.

Coherent configurations are a generalization of association schemes, and much of the theory carries over or can be modified. One association scheme idea that has not been considered is Delsarte's theory of subsets. Schrijver [10] has found new bounds on codes by considering subsets of the Terwilliger algebra of the Hamming scheme. Essentially, the bounds use a subset of the coherent configuration whose fibers are the weight of the words. This motivated our investigation of subsets of general coherent configurations.

We begin by defining coherent configurations. We use the notation of [8]; see that paper for more details than are given here.

Let X be a finite set, and let $\{f_i\}_{i\in I}$ be a set of relations on X partitioning $X \times X$ so that:

(1) $f_i \cap \operatorname{diag}(X \times X) \neq \emptyset$ implies $f_i \subseteq \operatorname{diag}(X \times X)$;
(2) given $i \in I$, $(f_i)^t = f_{i^*}$ for some $i^* \in I$, where $(f_i)^t = \{(y,x) : (x,y) \in f_i\}$;
(3) given $(x,y) \in f_k$, $|\{z : (x,z) \in f_i, (z,y) \in f_j\}|$ is a constant p^k_{ij} depending only on i, j, and k.

Then $\mathcal{C} = (X, (f_i)_{i\in I})$ is said to be a *coherent configuration*.

If instead of (1) and (2) we have

(1′) $f_i = \operatorname{diag}(X \times X)$ for some $i \in I$ and
(2′) $(f_i)^t = f_i$ for all $i \in I$,

then $(X, (f_i)_{i\in I})$ is a (symmetric) association scheme.

In an association scheme, the identity is one of the relations defining the scheme, whereas in a coherent configuration the identity is a union of relations. Let $\Omega = \{\alpha \in I : f_\alpha \subseteq \operatorname{diag}(X \times X)\}$. The relations $\{f_\alpha\}_{\alpha\in\Omega}$ determine a partition of X into *fibers* X_α with $f_\alpha = \operatorname{diag}(X_\alpha \times X_\alpha)$. For each $i \in I$, we have $f_i \subseteq X_\alpha \times X_\beta$ for some $\alpha, \beta \in \Omega$.

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